ISTANBUL TECHNICAL UNIVERSITY ★ GRADUATE SCHOOL OF SCIENCE
ENGINEERING AND TECHNOLOGY

# REPRESENTATION THEORY OF THE SYMMETRIC GROUP

Ph.D. THESIS

Ayşın ERKAN GÜRSOY

Department of Mathematical Engineering

Mathematical Engineering Programme

APRIL 2017

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Thesis Advisor: Prof. Dr. Vahap ERDOĞDU
Co-advisor: Dr. Kürşat AKER

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***To my husband İlker and especially my precious Zeynep,***

## FOREWORD

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APRIL 2017 Ayşın ERKAN GÜRSOY

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# REPRESENTATION THEORY OF THE SYMMETRIC GROUP

## SUMMARY

Polynomials are an age-old subject. Defined in terms of a base ring and finitely many indeterminates, polynomials are obtained by using only a finite number of additions and multiplications of the elements of the base ring and the indeterminates.

Symmetry is a fundamental concept of mathematics. At the heart of symmetry one wants to understand what remains invariant under a group action. Symmetric polynomials are the polynomials which are invariant under the permutation action of the symmetric group.

Catalan numbers is a sequence of positive integers which enumerate seemingly unrelated combinatorial objects.

This thesis is devoted to two subjects. First is the verification of an algorithm to calculate the expansion coefficients of a product of a Jack polynomial $J_\mu$ and a power-sum symmetric polynomial $p_n$ in the Jack polynomial basis for various partitions $\mu$ and integers $n$. The second is a new bijection between two sets of objects enumerated by Catalan numbers, solving an earlier conjecture in the paper "From Parking Functions to Gelfand Pairs" by K. Aker and M. B. Can in 2012.

Symmetric polynomials appear naturally in mathematics. For instance, the coefficients of a polynomial are symmetric polynomials of the roots of the given polynomials. These symmetric polynomials are called elementary symmetric polynomials, $e_n$. Summing up all monomials of a fixed total degree produces complete symmetric polynomials, $h_n$. Summing up the $n$-th power of indeterminates produces the power-sum symmetric polynomials, $p_n$. Symmetrizing monomials produces monomial symmetric polynomials, $m_\lambda$, where $\lambda$ is any partition. In fact, the monomial symmetric polynomials form a basis of the ring of symmetric polynomials. By extending the definitions of elementary, complete and power-sum symmetric polynomials appropriately, each of theses sets of polynomials constitute a basis for the symmetric polynomials.

In addition to these four bases, Schur polynomials, originally defined as the quotient of two determinants, form a fifth basis. Schur polynomials happen to be the most coveted basis among these five classical basis of symmetric polynomials. In representation theory, they correspond to the irreducible characters of the symmetric group. It is possible to study representation theory of symmetric groups by studying Schur polynomials.

When dealing with symmetric polynomials, it is best to work with infinitely many indeterminates to avoid accidental identities and to clarify the underlying structure. In terms of modern algebra, a symmetric function is an element of the symmetric function ring and the symmetric function ring is defined as the projective limit of rings of symmetric polynomials as the number of indeterminates tend to infinity.

Alternatively, a symmetric function can be thought of as a symmetric polynomial with finitely many indeterminates where the number of indeterminates can change. From this point of view, the crucial bit is that any identity involving symmetric functions must be valid as the number of indeterminates changes.

The product in the polynomials is so easy – the product of two monomials is again a monomial, add up the corresponding exponents – that we hardly think about it twice.

How about the product in the ring of symmetric functions? What are the structure constants for the ring of symmetric functions? This turns out to be either rather straightforward to be interesting as in the case of polynomials, or rather hard. For elementary, complete, power-sum symmetric functions, the answer is straightforward akin to the product of two monomials. The structure constants for the product of the two monomial symmetric functions is only obtained in 2001 in "A MAPLE program for calculations with Schur functions" by M.J. Carvalho, S. D'Agostino.

The structure constants for the product of two Schur functions is calculated by the Littlewood-Richardson rule in the paper "Group Characters and Algebra" by D. E. Littlewood and A. R. Richardson. First stated by D. E. Littlewood and A. R. Richardson in 1934, it took 4 decades for complete proofs to show up.

How about multiplying elements in different basis? How about multiplying Schur functions with other basis elements expanding in the Schur basis again?

The product of a Schur function with a complete symmetric function is calculated by the Pieri rule. The product of a Schur function with the power-sum symmetric functions is calculated by the Murnaghan-Nakayama rule and it calculates the values of the irreducible characters of the symmetric group.

Various other symmetric functions arose in mathematics in the twentieth century, such as Hall polynomials, zonal polynomials, Jack polynomials, Macdonald polynomials etc. They arose as the invariant polynomials for action of some group. For instance, zonal polynomials were first introduced in relation to orthogonal groups.

These functions can be seen as extensions of Schur functions. Schur functions, when expanded in terms of monomial functions, have triangular matrices. The set of Schur functions is orthonormal. When a normalization is imposed, Schur functions are uniquely defined by these properties. The above functions, albeit defined typically over a different base ring, can be characterized similarly. This is the approach followed in Macdonald's book.

In this thesis, our first consideration is to study the product of Jack polynomials with power-sum symmetric polynomials in terms of Jack polynomials. We search how to calculate the coefficients of this product and compute these coefficients step by step. Firstly, we prove that the product of an arbitrary Jack polynomial with a power sum symmetric function $p_1$ in the basis of Jack polynomials. Secondly, we prove that this product for the power sum symmetric function $p_2$. Then we find the formulas of the product of an arbitrary Jack polynomial with a power sum symmetric function for some cases. Thus the formula of the product $J_\mu\, p_n$ for any partition $\mu$ is generalized partially.

Catalan numbers pop up in all parts of mathematics. They enumerate a variety of different mathematical objects which seem unrelated at first impression. In the second problem, we prove a conjecture about the equality of two generating functions described in the paper "From Parking Functions to Gelfand Pairs" by K. Aker and M.

B. Can in 2012 attached to two sets whose cardinalities are given by Catalan numbers: We establish a new combinatorial bijection between the two sets which proves the equality of the corresponding generating functions.

# SİMETRİK GRUPLARIN TEMSİL TEORİSİ

## ÖZET

Polinomlar matematiğin en eski konularındandır. Bir temel halkaya ve sonlu sayıda değişkene göre tanımlanan polinomlar, bu değişkenlerin ve temel halkanın elemanlarının sonlu sayıda toplamları ve çarpımları kullanılarak elde edilir.

Simetri matematiğin temel kavramlarındandır. Simetrinin odak noktası bir grup etkisi altında değişmeden kalan yapıları anlamaktır. Simetrik grup, polinom halkası üzerinde değişkenler üzerinde permütasyonlarla etki eder. Simetrik grubun permütasyon etkisi altında değişmeyen polinomlarına simetrik polinomlar denir. İki simetrik polinomun toplamı ve çarpımı yine simetrik polinom olduğu için, simetrik polinomlar polinom halkasının bir alt-halkasını oluşturur.

Catalan sayıları matematikte birçok alanda karşımıza çıkan bir pozitif sayılar dizisidir. Catalan sayıları, ilk bakışta birbirleriyle ilgili olmayan pek çok kombinatorik kümeyi sayar.

Bu tez çalışması iki konuya ayrılmıştır. İlk konu, herhangi bir $\mu$ parçalanışı ve $n$ tamsayısı için herhangi bir Jack polinomu $J_\mu$ ile $n$-ninci kuvvet toplamı simetrik fonksiyonu $p_n$ çarpımının yine Jack polinomlarına göre açılmasından gelen genişletilmiş katsayıları hesaplayan algoritmanın sağlanması üzerinedir. İkinci konuda ise K. Aker ve M. B. Can'ın 2012 yılında yayınlanan makalelerinde yer alan bir iddia ispatlanmıştır. Bu iddia eleman sayıları Catalan sayıları olan iki küme arasında birebir ve örten yeni bir eşlemenin varlığı üzerinedir.

Simetrik polinomlar matematikte hemen her yerde karşımıza çıkar. Örneğin, bir polinomun katsayıları, polinomun köklerine göre simetrik polinomlardır. Bu simetrik polinomlara temel simetrik polinomlar denir; $e_n$ ile gösterilir. Derecesi aynı olan bütün tek terimlilerinin toplamı $h_n$ ile gösterilen tam simetrik polinomları üretir. $n$-ninci kuvvetten değişkenlerin toplamları, $p_n$ ile gösterilen kuvvet toplamı simetrik polinomları üretir. Simetrik hale getirilmiş tek terimliler, $\lambda$ herhangi bir parçalanış olmak üzere $m_\lambda$ ile gösterilen tek terimli simetrik polinomları üretir. Tek terimli simetrik polinomlar $m_\lambda$ simetrik fonksiyonlar halkasının bir bazıdır. Temel, tam ve kuvvet toplamı simetrik polinomlarının tanımlarının uygun olarak genişletilmesi ile bu polinomlar da simetrik polinomların birer bazını oluştururlar.

Bu dört baza ilave olarak Schur polinomları aslen iki determinantın bölümü olarak tanımlanan simetrik fonksiyonlar halkasının beşinci bazını oluşturur.

Simetrik fonksiyonlar uzayının doğrusal yapısı ile ilgili sorular bu önemli bazların kombinatorik olarak inşaasının temel noktalarıdır. Diğer bir odak noktası bu bazların oluşturduğu matrislerin değişimidir yani bu bazlar arasındaki geçiş matrislerini tarif etmektir. Baz matrislerinin değişimi kombinatoriğin doğası gereği zengin bilgiler içermektedir.

Simetrik polinomlar ile uğraşırken rastgele özdeşliklerden kaçınmak ve yapıyı açıklamak için sonsuz sayıda belirsizler ile çalışmak daha uygundur. Modern cebir diliyle simetrik fonksiyon, simetrik fonksiyon halkasının bir elemanıdır. Simetrik fonksiyon halkası ise belirsizlerinin sayısı sonsuza giden simetrik polinom halkalarının projektif limiti olarak tanımlanır.

Alternatif olarak simetrik fonksiyon, belirsizlerinin sayısının değişebildiği yerde sonlu sayıda belirsizlere sahip olan simetrik polinom olarak düşünülebilir. Bu açıdan bakıldığında önemli olan, simetrik fonksiyonlar arasındaki herhangi bir özdeşliğin belirsizlerin sayısı değiştikçe de geçerli olmasıdır.

Polinomların çarpımı oldukça kolaydır; iki tek terimlinin çarpımı yine bir tek terimlidir, karşılık gelen kuvvetlerin toplanması ile bulunur.

Simetrik fonksiyonlar halkasındaki çarpım nasıl olmalıdır? Simetrik fonksiyonlar halkası için yapı sabitleri nelerdir? Bu problemin çözümü ya ilgi çekici olmayacak kadar basit ya da hayli zordur. Temel simetrik fonksiyonlar, tam simetrik fonksiyonlar, kuvvet toplamı simetrik fonksiyonları için iki tek terimlinin çarpımı benzer şekilde açıktır.

Tek terimli simetrik polinomlar için yapı sabitleri, 2001 yılında M.J. Carvalho ve S. D'Agostino tarafında çözülmüştür.

Schur polinomlarının çarpımı, Littlewood-Richardson kuralı, D. E. Littlewood ve A. R. Richardson tarafından 1934 yılında formüle edildi. Hatasız ispatların ortaya çıkması 40 yıl aldı.

Farklı tabanlardaki elemanların çarpımı nasıl olmalıdır? Schur fonksiyonlarının diğer baz elemanları ile çarpımının yine Schur tabanında nasıl açılır?

Schur fonksiyonunun tam simetrik fonksiyon ile çarpılması Pieri kuralı ile hesaplanır. Schur fonksiyonunun kuvvet toplamı simetrik fonksiyonu ile çarpılması Murnaghan-Nakayama kuralı ile hesaplanır. Bu kural aslında simetrik grubun indirgenemez karakterlerinin değerlerini hesaplar.

Yirminci yüzyılda matematikte Hall polinomları, Zonal polinomlar, Jack polinomları, Macdonald polinomları gibi çeşitli başka simetrik fonksiyonlar ortaya çıktı. Bunlar bazı grupların etkisine göre değişmez polinomlar olarak ortaya çıktılar. Örneğin, ilk olarak Zonal polinomlar ortogonal gruplar ile ilgili ortaya çıkan ortogonal fonksiyonlardır.

Macdonald polinomları gibi, simetrik foksiyonlar Schur fonksiyonlarının genişlemeleri olarak görülebilirler. Schur fonksiyonları, tek terimli fonksiyonlara göre açıldığında üçgensel matrislere sahiptirler. Schur fonksiyonları birbirine göre ortonormaldir. Bir normalizasyon seçilince, bu özellikler Schur fonksiyonlarını tümüyle belirler. Yukarıda bahsi geçen Macdonald polinomları gibi fonksiyonlar, simetrik fonksiyon halkasının taban halkasının ve simetrik fonksiyonlar üzerindeki iç çarpımın genişletilmesi suretiyle tanımlanabilirler. I. G. Macdonald'ın kitabında izlenen yaklaşım budur.

Simetrik fonksiyon halkası içerisindeki çarpımlara geri dönersek, Schur fonksiyonlarının tam simetrik fonksiyonlar, birbirleri ve kuvvet toplamı simetrik fonksiyonları ile çarpımları kısaca şöyledir:

$n$ pozitif bir tamsayı olmak üzere, Pieri kuralı; $n$ nin herhangi bir $\mu$ parçalanışı için keyfi bir Schur fonksiyonu ile $n$-ninci tam simetrik fonksiyonun çarpımının yine Schur fonksiyonları cinsinden tarif edilmesidir. Yada başka bir deyişle; Schur fonksiyonları için Pieri kuralı; $n$ ninci tam simetrik fonksiyon, $(n)$ parçalanışlı Schur fonksiyonuna eşit olduğundan, birbirleri cinsinden yazılabildiğinden, $n$ nin herhangi bir $\mu$ parçalanışı için keyfi bir Schur fonksiyonu ile $(n)$ parçalanışlı Schur fonksiyonunun çarpımının yine Schur fonksiyonları cinsinden ifade edilmesidir.

Littlewood-Richardson kuralı; $n$ nin herhangi iki farklı parçalanışı için keyfi iki Schur fonksiyonun çarpımının yine Schur fonksiyonları cinsinden yazılmasıdır.

Murnaghan-Nakayama kuralı; $n$ nin herhangi bir parçalanışı için keyfi bir Schur fonksiyonu ile $n$-ninci kuvvet toplamı simetrik fonksiyonunun çarpımının Schur fonksiyonları cinsinden tarifidir ki bu yapı bize simetrik grupların indirgenemez karakterlerinin değerlerinin hesabı için rekürsif bir işlem tanımlar.

Bu çarpımlardan yola çıkarak çalıştığımız ilk problemimiz, simetrik fonksiyonlar uzayının diğer bir önemli bazı olan Jack simetrik fonksiyonları için Murnaghan-Nakayama kuralını araştırmaktır. Yani $n$ pozitif bir tamsayı olmak üzere $n$ nin herhangi bir parçalanışı için keyfi bir Jack simetrik fonksiyonu ile $n$-ninci kuvvet toplamı simetrik fonksiyonunun çarpımı, Jack simetrik fonksiyonları cinsinden ifade edildiğinde katsayıların nasıl hesaplanacağı ve ispatın nasıl yapılacağı sorusu ile ilgilidir.

Jack simetrik fonksiyonları ilk olarak istatistikçi Henry Jack tarafından 1969 yılında tanımlandı. Jack simetrik polinomları, taban halkasının bir $\alpha$ parametresi ve simetrik polinomlar üzerindeki iç çarpımın uygun bir şekilde genişletilmesiyle, Schur fonksiyonlarına benzer şekilde tanımlanır. Jack, $\alpha = 1$ durumunda bu polinomların, Schur fonksiyonlarına indirgeneceğini gösterdi. Ayrıca $\alpha = 2$ durumunda yine Jack fonksiyonlarının zonal polinomları verdiği çıkarımında bulundu. 1974 yılında H. O. Foulkes, Jack simetrik fonksiyonlarının kombinatoryel yorumu ile ilgili sorular ile ilgilendi. 1987 yılında I. G. Macdonald ise Jack fonksiyonlarının özelliklerini, başlangıç noktasını 1977 yılında Sekiguchi'nin çalıştığı diferansiyel operatörleri alarak araştırdı, bir dizi önemli eşitliklerin varlığını gösterdi. 1989 yılında ise R. P. Stanley bu konuyu oldukça geliştirdi: Jack simetrik fonksiyonları için Pieri kuralını oluşturdu ve ispatladı. Stanley, ek olarak, Jack simetrik fonksiyonları için Littlewood-Richardson kuralı hakkında birtakım çıkarımlarda bulundu.

İlk problemimizde öncelikle R. P. Stanley'in bulduğu Jack simetrik fonksiyonlar için Pieri kuralı ve $n$-ninci Jack simetrik fonksiyonları ile $n$-ninci kuvvet toplamı simetrik fonksiyonları arasındaki ilişki göz önünde bulundurulmuştur. Ayrıca 2004 yılında R. Sakamoto, J. Shiraishi, D. Arnaudon, L. Frappat ve E. Ragoucy'nin yazmış olduğu bir makalede bulunan herhangi bir Jack simetrik fonksiyonu ile $n$-ninci kuvvet toplamı simetrik fonksiyonları çarpımının algoritması üzerinde çalışılmıştır. Bu algoritmayı kullanarak adım adım herhangi bir Jack simetrik fonksiyonu ile $n$-ninci kuvvet toplamı simetrik fonksiyonları çarpımı, Jack simetrik fonksiyonları cinsinden ifade edildiğinde katsayıların neler olduğu, nasıl hesaplanacağı araştırılmıştır. Ayrıca adım adım bu çarpımların ispatları, Jack simetrik fonksiyonları için Pieri kuralı ve Jack simetrik fonksiyonların özellikleri kullanılarak araştırılmıştır.

İkinci problemimizde özellikle kombinatorik matematikte çok önemli bir yeri olan 1838 yılında Eugène Charles Catalan tarafından bulunan Catalan sayıları üzerinde çalışılmıştır. Özel bir pozitif sayı dizisi olan Catalan sayılarının $n$-ninci terimi $n \geq 0$ için $C_n = \frac{1}{n+1}\binom{2n}{n}$ formülü ile bulunur. Bu problemde, K. Aker ve M. B. Can'ın 2012 yılında basılmış olan makalelerinde bulunan bir iddia ispatlanmıştır. Bu iddia, eleman sayıları Catalan sayıları olan iki kümenin üreten fonksiyonlarının eşitliği hakkındadır. Bu iddianın ispatı, üreten fonksiyonlar eşitliği, iki küme arasındaki birebir ve örten kombinatoryel bir eşleme kurularak yapılmıştır.

# 1. INTRODUCTION

Algebraic combinatorics studies the ring of symmetric functions $\Lambda$ and other combinatorially constructed algebras. The fundamental questions in this field are:

- What is the linear structure of the ring of symmetric functions?

  1. Can one construct combinatorially significant linear basis of the symmetric function ring $\Lambda$? The answer is affirmative. There are many well-known combinatorial basis of the symmetric function ring $\Lambda$, e.g. monomial functions, complete symmetric functions, power sum symmetric functions, Schur functions, Jack symmetric functions, Macdonald polynomials etc. Jack symmetric functions and Macdonald polynomials are generalizations of Schur symmetric functions with parameters.
  2. How are different basis related to each other? That is, what are the change of basis matrices? The entries of the matrices correspond to enumeration of certain sets.

- What is the nonlinear structure of the ring of symmetric functions $\Lambda$?

  3. How do two basis elements (possibly from different basis) multiply? Some famous product formulas are Littlewood-Richardson rule which calculates the product of two Schur functions (Section 1.3.8), Pieri rule which calculates the product of a Schur function with a complete symmetric function (Section 1.3.9) and Murnaghan-Nakayama rule which calculates the product of a Schur function with a power sum symmetric function in terms of Schur functions (Section 1.3.11).

## 1.1 Purpose of Thesis

In this thesis, we study two problems: The first is to find a Murnaghan-Nakayama type formula for the product of a Jack symmetric function and a power sum symmetric function. The second is to construct a new bijective proof for the celebrated Catalan numbers.

In the first problem, we study the product of an arbitrary Jack polynomial with an $n$-th power sum symmetric function in the basis of Jack polynomials. We understand how to calculate the coefficients of this product and compute these coefficients step by step.

The fundamental definitions and theorems which we would use would be given in Chapter 1. Basic definitions and theorems in Sections 1.3.1, 1.3.4 and 1.3.11, 1.3.12, 1.3.13 and 1.3.14 are taken verbatim from [1], [2], [3] and [4] respectively.

In Chapter 2, using the Pieri rule for Jack symmetric functions in Theorem 59 and Algorithm 61 in Section 1.3.14, we prove that the product of an arbitrary Jack polynomial with a power sum symmetric function $p_1$ in the basis of Jack polynomials in Theorem 62. Then the product of an arbitrary Jack polynomial with a power sum symmetric function $p_2$ in terms of Jack polynomials in Theorem 64 is proved. We study the coefficients of Jack polynomials obtained by the product of a Jack polynomial with a power sum symmetric function $p_1$ and $p_2$.

We find some formulas for the product $J_\mu \, p_n$ in some cases and hence some formula of for product $J_\mu \, p_n$ is generalized partially in Chapter 3.

In Chapter 4, our second problem related to the Catalan numbers is studied. This chapter is devoted my joint paper "A New Combinatorial Identity for Catalan Numbers" with Kursat Aker to appear in Ars Combinatoria. The rest of the chapter is essentially a verbatim copy of the said paper. We show the existence of a bijection between two sets whose cardinality is a Catalan number. Also, we prove that two generating functions derived from different incarnations of Catalan Numbers coincide. The generating function $\sum q^{\binom{n}{u}}$ appears in [5] as the dimension counting generating function for the parking function module. In [5], the authors conjecture that this generating function coincides with another generating function $\sum q^{\binom{n}{v}}$. We prove this conjecture by combinatorial arguments.

## 1.2 Literature Review

Jack's symmetric functions were first defined by the statistician Henry Jack in 1969 [6], [7]. He showed that when $\alpha = 1$ they reduce to the Schur functions, and conjectured that when $\alpha = 2$ they should give the zonal polynomials. Later, H. O. Foulkes [8] raised the question of finding a combinatorial interpretation of Jack's symmetric functions. He began to investigate their properties, taking the differential operators of Sekiguchi [9] (see Chapter $VI.3$, Example 3) as a starting point, and showed in particular [10] that they satisfy $(10.13), (10.14),$ and $(10.17)$ in [3] (duality). Shortly afterwards R. P. Stanley [4] advanced the subject further, and established the scalar product formula $(10.16)$ in [3], the Pieri formula, the explicit expression as a sum over tableaux, and the specialization Theorem $(10.20)$ in [3]. Stanley worked with the Jack symmetric function $J_\lambda^\alpha$ rather than $P_\lambda^\alpha$.

Sakamoto et al. [11] study the action of the Virasoro operators $L_n$ on the Fock space. They give an algorithm to calculate the matrices of a Virasoro operators with respect to Jack basis.

The Catalan numbers form a sequence of nonnegative numbers that appear in various counting problems in combinatorics. Catalan numbers are called after the Belgian mathematician Eugène Charles Catalan (1814-1894). R. P. Stanley has listed over seventy existences of Catalan numbers in his book [1] and seventy more in his web site Catalan Addendum [12].

### 1.3 Background

Partitions (Definition 1) enumerate many combinatorial objects as well as different basis of symmetric functions. To clarify how the symmetric functions basis are enumerated by partitions, first think of, the ring of polynomials (possibly in infinite number of indeterminates). In contrast, a basis of ring of polynomials will be enumerated by finite sequences of nonnegative integers. Each orbit of finite sequences of nonnegative integers under the action of symmetric group contains a unique partition. This is why symmetric function basis are enumerated by partitions. The graphical depiction of a partition is called a Young diagram (Definition 4).

Young diagrams enrich the representations of partitions by transforming 1-dimensional partitions into 2-dimensional objects. Young diagrams further can be enriched by allowing boxes to be filled with nonnegative integers with certain restrictions. The resulting objects are called Young Tableaux (Definition 10).

We introduce matrix representations (Definition 18) and group characters of symmetric groups (Definition 23) to present the application of Murnaghan-Nakayama rule (Theorem 47) to calculate irreducible symmetric group characters (Theorem 48).

Following Stanley, we first introduce monomial symmetric functions $m_\lambda$ (Definition 31), then we introduce the symmetric function ring $\Lambda$ spanned by monomial symmetric functions.

For positive integer $n$, the symmetric group $S_n$ acts on the polynomial ring $\mathbb{Q}[x_1, x_2, \ldots, x_n]$ by permuting the indices. The ring of symmetric polynomials $\Lambda_n$ is the ring of invariants $\mathbb{Q}[x_1, x_2, \ldots, x_n]^{S_n}$. The symmetric polynomial rings $\Lambda_n$ form a projective system, where $\Lambda_{n+1} \to \Lambda_n$ is a projection defined by $x_i \mapsto x_i$ for $i = 1, \ldots, n$ and $x_{n+1} \mapsto 0$. The symmetric function ring $\Lambda$ is the projective limit of the symmetric polynomials rings $\Lambda_n$.

We introduce the other fundamental basis for the symmetric function ring $\Lambda$, namely the power sum symmetric functions $p_\lambda$, the elementary symmetric functions $e_\lambda$ and the complete symmetric functions $h_\lambda$.

Another cornerstone of the theory of symmetric functions is the inner product on the symmetric function ring $\Lambda$. It suffices to define an inner product on any basis. The inner product on the ring of symmetric functions is defined so that the power sum symmetric functions are orthogonal. For partitions $\lambda$ and $\mu$, set

$$\langle p_\lambda, p_\mu \rangle = \delta_{\lambda\mu} z_\lambda.$$

Here, $z_\lambda = \prod_{i \geq 1} i^{m_i} . m_i!$ where $m_i = m_i(\lambda)$ is the number of parts of $\lambda$ equal to $i$.

In Section 1.3.7, we introduce Schur functions in terms of Young Tableaux (Section 1.3.2) and touch upon their most important properties.

Schur basis can be simply characterized as an orthonormal basis, which is triangular with respect to the monomial basis, plus a normalization condition.

After this brief review of the linear structure of the symmetric function ring $\Lambda$, we finally discuss some well-known product formulas. Littlewood-Richardson rule for the product of two Schur functions (Section 1.3.8) and its special case, Pieri rule for the product of a Schur function and a complete symmetric function (Section 1.3.9).

In Section 1.3.11, we introduce the Murnaghan-Nakayama rule which calculates the product of an arbitrary Schur function $s_\lambda$ with a power sum symmetric function $p_n$ expanding the product in Schur functions (Theorem 47). A direct application of this theorem would give us irreducible character values for symmetric groups (Theorem 48).

Schur functions can be extended in many directions. Two extensions within the realm of symmetric functions are Macdonald polynomials and Jack polynomials. These extensions require extending the base field by parameters $q, t$ in the case of Macdonald polynomials and by a parameter $\alpha$ in the case of Jack polynomials.

To define Macdonald and Jack polynomials, one deforms the inner product $\langle \cdot, \cdot \rangle$ properly. Recall that Schur basis can defined as the orthonormal, triangular basis with respect to the monomial basis. Using this characterization as a guide, Macdonald and Jack polynomials can be defined as the unique orthonormal basis, triangular with respect to the monomial basis which satisfy a normalization condition for the corresponding inner product.

Macdonald polynomials specialize to Jack polynomials under a certain limit and Jack polynomials specialize to Schur functions by setting $\alpha$ to 1.

In Section 1.3.12, we touch up on the Macdonald polynomials. Later in Sec. 1.3.13, we describe the fundamental properties of Jack polynomials as will be needed in this thesis, followed by Stanley's theorem for a Pieri rule for Jack polynomials and Sakamoto's Algorithm (Section 1.3.14).

### 1.3.1 Partitions

**Definition 1.** A partition $\lambda$ of a nonnegative integer $n$ is a sequence $(\lambda_1, ..., \lambda_k) \in \mathbb{N}^k$ satisfying $\lambda_1 \geq ... \geq \lambda_k$ and $\sum \lambda_i = n$. Any $\lambda_i = 0$ is considered irrelevant, and we identify $\lambda$ with the infinite sequence $(\lambda_1, ..., \lambda_k, 0, 0, ...)$. We let $\mathrm{Par}(n)$ denote the set of all partitions of $n$, with $\mathrm{Par}(0)$ consisting of the empty partition $\emptyset$ (or the sequence $(0, 0, ...)$), and we let

$$Par := \bigcup_{n \geq 0} Par(n).$$

**Example 2.** Let the partition 3221 as short for $(3,2,2,1,0,...)$. Then we write

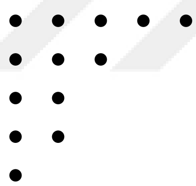
$$
\begin{aligned}
Par(1) &= \{1\} \\
Par(2) &= \{2,11\} \\
Par(3) &= \{3,21,111\} \\
Par(4) &= \{4,31,22,211,1111\}.
\end{aligned}
$$

**Definition 3.** If $\lambda \in Par(n)$, we also write $\lambda \vdash n$ or $\mid \lambda \mid = n$. The number of parts of $\lambda$ (i.e., the number of nonzero $\lambda_i$) is the length of $\lambda$, denoted $\ell(\lambda)$.

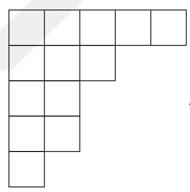
**Definition 4.** Suppose $\lambda = (\lambda_1, \lambda_2, ..., \lambda_r) \vdash n$. The *Young diagram* (also called a *Ferrers diagram*) of a partition $\lambda$ is an array of $n$ dots with $r$ left-justified rows in which row $i$ contains $\lambda_i$ dots for $1 \leq i \leq r$.

The dot in row $i$ and column $j$ has coordinates $(i,j)$, as in a matrix. Boxes (also called *cells*) are often used in place of dots.

**Example 5.** The Young diagram (Ferrers diagram) of the partition $\lambda = (5,3,2,2,1)$ is

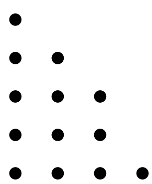


*or*

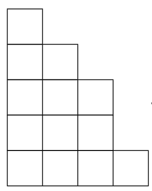


.

This is the English notation. The French notation uses the Cartesian coordinate system with the usual origin and the $x$ and $y$ directions:



*or*



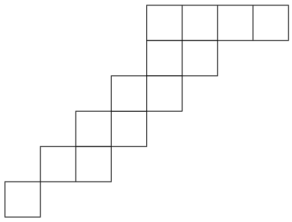
.

We will use the English notation. A partition $\lambda$ and its shape will be identified throughout this work.

Let $\lambda$ and $\mu$ be two partitions. Then we write $\mu \subseteq \lambda$ if $\mu_i \leq \lambda_i$ for all $i \geq 1$.

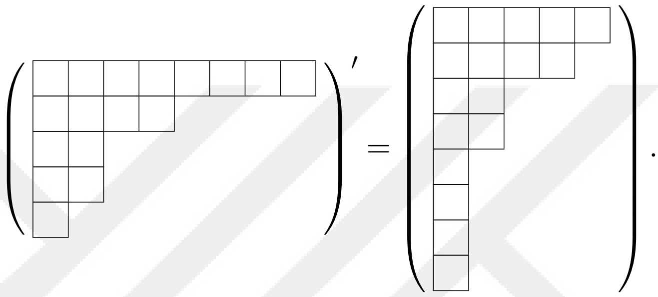
When $\mu \subseteq \lambda$, the diagram obtained by removing the cells of $\mu$ from the cells of $\lambda$ is called a skew diagram and is denoted by $\lambda/\mu$.

**Example 6.** The Young diagram for the skew shape $(8,6,5,4,3,1)/(4,4,3,2,1)$ is



**Definition 7.** The *transpose* of a partition $\lambda$, denoted by $\lambda'$, is the reflection of the Young diagram of $\lambda$ with respect to the main diagonal, $i = j$.

**Example 8.** The transpose of the partition $(8,4,2,2,1)$ is $(5,4,2,2,1,1,1,1)$:



For fixed $n$, define a total ordering on $Par(n)$, called *lexicographic ordering*: For $\lambda, \mu \in Par(n)$, then $\lambda <_L \mu$ if there exists a positive integer $r$ such that

1. $\lambda_i = \mu_i$ *for* $i < r$ *and*

2. $\lambda_r < \mu_r$.

There is also a commonly used partial ordering on the set of partitions, called *dominance order*, and denoted by $<_D$, or just $<$. For $\lambda, \mu \in Par(n)$, then $\lambda <_D \mu$ if for all positive integers $r$, the following holds:

$$\sum_{i=1}^{r} \lambda_i \leq \sum_{i=1}^{r} \mu_i.$$

It is easy to see that the lexicographic order refines the dominance order. We also have the following standard fact, the proof of which can be found in [11].

**Proposition 9.** *Transposition reverses dominance order. That is, $\lambda \leq_D \mu$ if and only if $\mu' \leq_D \lambda'$.*

### 1.3.2 Tableaux

**Definition 10.** A *Young tableau* $T$ is obtained by filling the integers $1,2,...,n$ in the $n$ boxes of the Young diagram of $\lambda$.

The underlying partition of a Young tableau $T$ is called the *shape* of the Young tableau $T$ which we denote by $sh(T)$.

**Definition 11.** A Young tableau $T$ of shape $\lambda$ is *standard* if each rows and each columns are strictly increasing sequences.

**Example 12.** The tableau

$$T = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array}$$

is standard, whereas

$$T = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 5 & 2 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array}$$

is not.

**Definition 13.** A Young tableau $T$ of shape $\lambda$ is *semi-standard* if each row weakly increases and each column strictly increases.

**Example 14.**

$$T = \begin{array}{|c|c|c|}\hline 2 & 2 & 3 \\ \hline 3 & 4 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array}$$

is semi-standard, but the tableau

$$T = \begin{array}{|c|c|c|}\hline 3 & 2 & 2 \\ \hline 4 & 3 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array}$$

is not.

A Young tableau $T$ has type $\alpha = (\alpha_1, \alpha_2, ....)$ where $\alpha_i = \alpha_i(T)$ is the multiplicity $i$ in tableau $T$. For any Young tableau $T$ of type $\alpha$, let

$$x^T = \prod_{i \geq 1} x_i^{\alpha_i(T)}.$$

**Example 15.** Let

$$T = \begin{ytableau} 1 & 2 & 2 \\ 3 & 5 \\ 6 \end{ytableau}\ .$$

As is seen, $T$ is a semi-standard tableau with

1. $|T| = 6$,

2. $sh(T) = (3,2,1)$,

3. $\alpha = (1,2,1,0,1,1,0,0,...)$,

4. $x^T = x_1x_2^2x_3x_5x_6$.

The symmetric group $S_n$ acts on tableaux by permuting the entries: For $\pi \in S_n$ acts on a tableau $T = (T(i,j))$ of shape $\lambda \vdash n$ by

$$\pi T = (\pi(T(i,j))).$$

**Example 16.** If $\pi = (1,2,3)$ and $T = \begin{ytableau} 1 & 3 \\ 2 \end{ytableau}$, then

$$(1,2,3)\begin{ytableau} 1 & 3 \\ 2 \end{ytableau} = \begin{ytableau} 2 & 1 \\ 3 \end{ytableau}.$$

### 1.3.3 Compositions

**Definition 17.** A *composition* of a nonnegative integer $n$ is a vector of positive integers $\alpha = (\alpha_1, \alpha_2, ..., \alpha_r)$ such that $\sum_{i=1}^r \alpha_i = n$. We write $\alpha \models n$ to denote $\alpha$ is a composition of $n$. The integer $r$ is called the length of $\alpha$ and denoted by $\ell(\alpha)$. Similarly, the integer $n$ is called the size of $\alpha$ and is denoted by $|\alpha|$. The integers $\alpha_i$ are called the *parts* of $\alpha$. We write *Comp(n)* for the set of compositions of $n$.

### 1.3.4 Matrix Representations

Groups are abstract objects. It would be convenient if we could always put them in some standard, equivalent form. Forming a matrix representation of an abstract group can be thought of as a process of transforming the abstract group into a concrete group of matrices.

Let $\mathrm{Mat}_d$ stand for the set of all $d \times d$ matrices with entries in the complex numbers $\mathbb{C}$. The vector space $\mathrm{Mat}_d$ is called the *full complex matrix algebra of degree d*.

Recall that an algebra is a vector space with an associative multiplication of vectors (hence also imposing a ring structure on the space). The *complex general linear group of degree d*, denoted by $GL_d$, is the group of all invertible matrices $\rho = (x_{i,j})_{d\times d} \in \mathrm{Mat}_d$.

**Definition 18.** A *matrix representation of a group G* is a group homomorphism

$$\rho : G \to GL_d.$$

Equivalently, to each $g \in G$ is assigned $\rho(g) \in \mathrm{Mat}_d$ such that

1. $\rho(e) = I$ the identity matrix, and
2. $\rho(gh) = \rho(g)\rho(h)$ for all $g,h \in G$.

The parameter $d$ is called the *degree*, or *dimension*, of the representation and is denoted by $\deg\rho$.

Evidently the simplest representations are those of degree 1.

**Example 19.** All groups have the *trivial representation*, which assigns the elements matrix $(1)$ to all $g \in G$. This is clearly a representation because $\rho(e) = (1)$ and

$$\rho(g)\rho(h) = (1)(1) = (1) = \rho(gh)$$

for all $g,h \in G$. We often use $1_G$ or simply 1 to denote the trivial representation of $G$.

**Example 20.** For the symmetric group $S_n$, the sign of a permutation, denoted $sgn(\pi)$ for $\pi \in S_n$, defines a 1-dimensional representation. The resulting representation is called the *sign representation*.

**Example 21.** Also of importance is the *defining representation* of $S_n$, which is of degree $n$. For $\pi \in S_n$, let $\rho(\pi) = (x_{i,j})_{n\times n}$, where

$$x_{i,j} = \begin{cases} 1, & \text{if } \pi(j) = i \\ 0, & \text{otherwise} \end{cases}$$

The resulting matrices $\rho(\pi)$ are called the *permutation matrices*.

**Example 22.** For the symmetric group on 3-letters, here are the resulting permutation matrices. The permutations are written in cycle notation:

$$\rho(\varepsilon) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \rho((1,2)) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\rho((1,3)) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \rho((2,3)) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\rho((1,2,3)) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \rho((1,3,2)) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

### 1.3.5 Group Characters

**Definition 23.** Let $\rho : G \to GL_d$ be a matrix representation. Then the *character of* $\rho$ is

$$\chi^{\rho}(g) = \mathrm{Tr}\,\rho(g),$$

where Tr denotes the trace of a matrix and the character is usually denoted with the Greek letter $\chi$. Otherwise put, $\chi^{\rho}$ is the map

$$\chi^{\rho} : G \overset{\mathrm{Tr}\rho}{\to} \mathbb{C}.$$

**Definition 24.** Let $V$ be a vector space and $G$ be a group. Then $V$ is a *G-module* if there is a group homomorphism

$$\rho : G \to GL(V).$$

Equivalently, $V$ is a *G-module* if there is a multiplication, $g\mathbf{v}$, of elements of $V$ by elements of $G$ such that

1. $g\mathbf{v} \in V$,

2. $g(c\mathbf{v} + d\mathbf{w}) = c(g\mathbf{v}) + d(g\mathbf{w})$,

3. $(gh)\mathbf{v} = g(h\mathbf{v})$,

4. $e\mathbf{v} = \mathbf{v}$

for all $g, h \in G$; $\mathbf{v}$, $\mathbf{w} \in V$; and scalars $c, d \in \mathbb{C}$.

If $V$ is a G-module, then its *character* is the character of a matrix representation $\rho$ corresponding to $V$.

Note that if $X$ is any set with a multiplication by elements of $G$ satisfying $(1),(3)$ and $(4)$, then we say *$G$ acts on $X$*.

Since there are many matrix representations corresponding to a single $G$-module, we should check that the module character is well-defined.

**Definition 25.** Let $V$ and $W$ be $G$-modules. Then a *$G$-homomorphism* (or simply a *homomorphism*) is a linear transformation $\theta : V \to W$ such that

$$\theta(gv) = g\theta(v)$$

for all $g \in G$ and $v \in V$. We also say that $\theta$ preserves or respects the action of $G$.

**Definition 26.** Let $V$ and $W$ be modules for a group $G$. A *$G$-isomorphism* is a $G$-homomorphism $\theta : V \to W$ that is bijective. In this case we say that $V$ and $W$ are *$G$-isomorphic*, or *$G$-equivalent*, written $V \cong W$.

If $\rho$ and $\phi$ both correspond to $V$, then $\phi = A\rho A^{-1}$ for some fixed $A$. Hence, for all $g \in G$,

$$\mathrm{Tr}\,\phi(g) = \mathrm{Tr} A\,\rho(g)A^{-1} = \mathrm{Tr}\,\rho(g),$$

since trace is invariant under conjugation. Hence $\rho$ and $\phi$ have the same character.

For a given group $G$ and a degree 1 representation, the character of the representation and the representation itself can be identified. Therefore,

**Example 27.** The character corresponding to a degree 1 representation is called a *linear character*.

**Example 28.** Consider the defining representation of $S_n$ with its character $\chi^{\mathrm{def}}$. If $n = 3$, then we can compute the character values by direct calculation using the permutation matrices in Example 22:

$$\chi^{\mathrm{def}}(\varepsilon) = 3, \quad \chi^{\mathrm{def}}((1,2)) = 1, \quad \chi^{\mathrm{def}}((1,3)) = 1,$$

$$\chi^{\mathrm{def}}((2,3)) = 1, \quad \chi^{\mathrm{def}}((1,2,3)) = 0, \quad \chi^{\mathrm{def}}((1,3,2)) = 0.$$

In general, if $\pi \in S_n$, then

$$\begin{aligned}\chi^{\text{def}}(\pi) &= \text{the number of ones on the diagonal of } \rho(\pi)\\ &= \text{the number of fixed points of } \pi.\end{aligned}$$

**Proposition 29.** *Let $\rho$ be a matrix representation of a group $G$ of degree $d$ with its character $\chi^\rho$.*

*1. $\chi^\rho(e) = d$.*

*2. If $H$ is a conjugacy class of $G$, then*

$$g,h \in H \Rightarrow \chi^\rho(g) = \chi^\rho(h).$$

*3. If $\phi$ is a representation of $G$ with character $\psi$, then*

$$\rho \cong \phi \Rightarrow \chi^\rho(g) = \psi^\phi(g)$$

*for all $g \in G$.*

### 1.3.6 The Ring of Symmetric Functions

Let $x = \{x_1, x_2, x_3, \ldots\}$ be an infinite set of variables and consider the formal power series ring $\mathbb{Q}[[x]]$. A monomial $x_{i_1}^{\lambda_1} x_{i_2}^{\lambda_2} \ldots x_{i_r}^{\lambda_r}$ is said to have *degree* $\sum_i \lambda_i$.

For every positive integer $n$, the symmetric group $S_n$ acts on the power series ring $\mathbb{Q}[[x]]$ naturally, for $\pi \in S_n$ and $f(x) \in \mathbb{Q}[[x]]$, define the action by

$$\pi f(x_1, x_2, x_3, \ldots) = f(x_{\pi 1}, x_{\pi 2}, x_{\pi 3}, \ldots),$$

where $\pi i = i$ for $i > n$.

One can produce symmetric functions by symmetrizing a given monomial.

**Definition 30.** [1] Let $x = (x_1, x_2, \ldots)$ be a set of indeterminates, and let $n \in \mathbb{N}$. A homogeneous symmetric function of degree $n$ over a commutative ring $R$ (with identity) is a formal power series

$$f(x) = \sum_{\alpha} c_\alpha x^\alpha,$$

where (a) $\alpha$ ranges over all weak compositions $\alpha = (\alpha_1, \alpha_2, \ldots)$ of $n$ (of infinite length), (b) $c_\alpha \in R$, (c) $x^\alpha$ stands for the monomial $x_1^{\alpha_1} x_2^{\alpha_2} \ldots$, and (d) $f(x_{w(1)}, x_{w(2)}, \ldots) = f(x_1, x_2, \ldots)$ for every permutation $w$ of the positive integers $\mathbb{P}$.

**Definition 31.** Let $\lambda = (\lambda_1, \lambda_2, ..., \lambda_r)$ be a partition. The *monomial symmetric function corresponding to* $\lambda$ is

$$m_\lambda = m_\lambda(x) = \sum x_{i_1}^{\lambda_1} x_{i_2}^{\lambda_2} ... x_{i_r}^{\lambda_r},$$

where the sum is over all distinct monomials with exponents $\lambda_1, \lambda_2, ..., \lambda_r$.

**Example 32.** If $\lambda = (2,1)$ then the monomial symmetric function corresponding to $\lambda$ is

$$m_{(2,1)} = x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2 + ...$$

Obviously, if $\lambda \vdash n$, then $m_\lambda(x)$ is homogeneous of degree $n$.

**Definition 33.** The *ring of symmetric functions* is

$$\Lambda = \Lambda(x) = \mathbb{Q}m_\lambda,$$

that is, the vector space spanned by all the $m_\lambda$.

The ring of symmetric functions $\Lambda$ is in fact a graded algebra with respect to degree:

$$\Lambda = \bigoplus_{n \geq 0} \Lambda^n.$$

Here, the graded piece $\Lambda^n$ is the vector space spanned by all $m_\lambda$ of degree $n$.

Since the $m_\lambda$ are independent,

**Proposition 34.** *Monomial symmetric functions $m_\lambda$, $\lambda \vdash n$ form a basis of the homogeneous symmetric functions for the space $\Lambda^n$.*

*Its dimension equals the number of partitions of n.*

**Definition 35.** The *nth power sum symmetric function* is

$$p_n = m_{(n)} = \sum_{i \geq 1} x_i^n.$$

The *nth elementary symmetric function* is

$$e_n = m_{(1^n)} = \sum_{i_1 < ... < i_n} x_{i_1} ... x_{i_n}.$$

The *nth complete homogeneous symmetric function* is

$$h_n = \sum_{\lambda \vdash n} m_\lambda = \sum_{i_1 \leq ... \leq i_n} x_{i_1} ... x_{i_n}.$$

For a given partition $\lambda$, define the power sum symmetric function $p_\lambda$ as

$$p_\lambda = \prod_i p_{\lambda_i}.$$

The elementary symmetric function $e_\lambda$ and the complete symmetric function $h_\lambda$ for an arbitrary partition $\lambda$ are defined analogously.

**Example 36.** If $n = 3$ then we have

$$\begin{aligned}
p_3 &= x_1^3 + x_2^3 + x_3^3 + \ldots, \\
e_3 &= x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4 + \ldots, \\
h_3 &= x_1^3 + x_2^3 + \ldots + x_1^2x_2 + x_1x_2^2 + \ldots + x_1x_2x_3 + x_1x_2x_4 + \ldots.
\end{aligned}$$

**Theorem 37.** *The following are bases for $\Lambda^n$.*

*1.* $\{p_\lambda : \lambda \vdash n\}$.

*2.* $\{e_\lambda : \lambda \vdash n\}$.

*3.* $\{h_\lambda : \lambda \vdash n\}$.

**Example 38.** If $\lambda = (2,1)$ then we obtain

$$\begin{aligned}
p_{(2,1)} &= p_2p_1 = (x_1^2 + x_2^2 + x_3^2 + \ldots)(x_1 + x_2 + x_3 + \ldots), \\
e_{(2,1)} &= e_2e_1 = (x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + \ldots)(x_1 + x_2 + x_3 + \ldots), \\
h_{(2,1)} &= h_2h_1 = (x_1^2 + x_2^2 + x_3^2 + \ldots + x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + \ldots)(x_1 + x_2 + x_3 + \ldots).
\end{aligned}$$

Denote the multiplicity of the part $i$ in the partition $\lambda$ by $m_i(\lambda)$. Let $z_\lambda = \prod_{i \geq 1} i^{m_i}.m_i!$ where $m_i = m_i(\lambda)$.

Define the inner product $\langle \cdot, \cdot \rangle$ on the symmetric function ring $\Lambda$ as follows:

**Definition 39.** For partitions $\lambda$ and $\mu$, set

$$\langle p_\lambda, p_\mu \rangle = \delta_{\lambda\mu} z_\lambda.$$

As power sum symmetric functions form a basis of the ring of symmetric functions, by linear extension, this definition determines the inner product completely.

### 1.3.7 Schur Functions

Schur functions $s_\lambda$ constitute a basis for the ring of symmetric functions, $\Lambda^n$. As we will see, they are also intimately connected with the irreducible representations of the symmetric groups $S_n$ and the Young tableaux. In fact, there are several ways to define Schur functions. We follow a combinatorial approach.

**Definition 40.** Let $\lambda$ be a partition. The associated *Schur function* is

$$s_\lambda(x) = \sum_T x^T,$$

where the sum is over all semi-standard $\lambda$ -tableaux $T$.

**Example 41.** If $\lambda = (2,1)$, then some of the possible tableaux are

$$T : \ytableaushort{11,2}\,, \ \ytableaushort{12,2}\,, \ \ytableaushort{11,3}\,, \ \ytableaushort{13,3}\,, \ \ldots, \ \ytableaushort{12,3}\,, \ \ytableaushort{13,2}\,, \ \ytableaushort{12,4}\,, \ \ytableaushort{14,2}\,, \ \ldots,$$

so

$$s_{(2,1)}(x) = x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + x_1x_3^2 + \ldots + 2x_1x_2x_3 + 2x_1x_2x_4 + \ldots.$$

Note that if $\lambda = (n)$, then one-rowed tableau is just a weakly increasing sequence of $n$ positive integers, that is, a partition with $n$ parts so

$$s_{(n)}(x) = \sum_{i_1 \leq \ldots \leq i_n} x_{i_1} \ldots x_{i_n} = h_n(x).$$

If $\lambda$ consists of a single column, i.e. $\lambda = (1^n)$, then the entries must increase from top to bottom, so the partition must have distinct parts and hence

$$s_{(1^n)}(x) = \sum_{i_1 < \ldots < i_n} x_{i_1} \ldots x_{i_n} = e_n(x).$$

**Proposition 42.** *The function $s_\lambda(x)$ is symmetric.*

**Definition 43.** For a given partition $\lambda$ and a composition $\mu$, the number of semi-standard Young tableaux of shape $\lambda$ and type $\mu$ is called the Kostka number $K_{\lambda\mu}$.

Recall that $\mu \leq_D \lambda$ is the dominance order on the set of partitions.

**Proposition 44.** *The change of basis matrix from monomial symmetric functions to Schur functions is triangular with trivial diagonal: For a given partition $\lambda$, the Schur function*

$$s_\lambda = \sum_{\mu \leq_D \lambda} K_{\lambda\mu} m_\mu, \tag{1.1}$$

*where the sum is over partitions $\mu$. Here,*

$$K_{\lambda\mu} = \begin{cases} 0, & \mu \not\leq_D \lambda \\ 1, & \lambda = \mu. \end{cases}$$

Notice that although the Kostka number $K_{\lambda\mu}$ requires a partition $\lambda$ and composition $\mu$, in the above theorem $\mu$ denotes a partition.

Schur functions are triangular with respect to monomial symmetric functions:

**Definition 45.** When basis elements $b_\lambda$ of the symmetric function ring $\lambda$ satisfies equations akin to 1.1, we will say that the basis $b_\lambda$ is triangular with respect to monomial symmetric functions.

Schur functions constitute the unique basis $b_\lambda$ so that

1. The basis $b_\lambda$ is orthonormal with respect to the inner product $\langle \cdot, \cdot \rangle$: That is, $\langle b_\lambda, b_\mu \rangle = \delta_{\lambda\mu}$.
2. The basis $b_\lambda$ is triangular with respect monomial symmetric functions.
3. The coefficient of $m_\lambda$ in the expansion of $b_\lambda$ equals 1.

**Corollary 46.** *The Schur functions $s_\lambda, \lambda \vdash n$ form a orthonomal basis for $\Lambda^n$.*

### 1.3.8 Littlewood-Richardson Rule

The multiplicative structure of the ring $\Lambda$ is determined by Littlewood-Richardson formula: For partitions $\lambda, \mu$,

$$s_\lambda s_\mu = \sum_\nu c^\nu_{\lambda\mu} s_\nu,$$

where the sum runs over partitions $\nu$ and the structure constants $c^\nu_{\lambda\mu}$ count the number of semi-standard skew tableau of shape $\nu/\lambda$ and weight $\mu$. The structure constants $c^\nu_{\lambda\mu}$ are called the *Littlewood-Richardson coefficients*.

### 1.3.9 Pieri Rule

A special case of the Littlewood-Richardson rule is the Pieri rule, which calculates the product of the Schur function $s_\lambda$ and the complete symmetric function $h_n$. Recall that $h_n$ is the Schur function $s_{(n)}$. Then, for any partition $\lambda$, the Pieri rule states that

$$s_\lambda h_n = \sum_\nu s_\nu,$$

where the sum runs of over all partitions $\nu$ so that $\nu/\lambda$ is a horizontal $n$-strip. A skew-shape is a horizontal strip if each column contains at most one box.

### 1.3.10 Dual Pieri Rule

Dual Pieri rule computes the product of a Schur function $s_\lambda$ with an elementary symmetric function $e_n$. Recall that $e_n$ is the Schur function $s_{(1^n)}$. The dual Pieri rule states that

$$s_\lambda e_n = \sum_\nu s_\nu,$$

where the sum runs over all partitions $\lambda \subset \nu$ such that $\nu/\lambda$ is a vertical $n$-strip. A skew-diagram is a vertical strip if each row contains at most one box.

### 1.3.11 Murnaghan-Nakayama Rule

Given a partition $\lambda$ and a nonnegative integer $n$, Murnaghan-Nakayama rule calculates the product of a Schur function $s_\lambda$ and a power symmetric function $p_n$.

A skew-diagram is called a *border strip* if it is connected and contains no $2 \times 2$ block of squares. In literature, terms *skew hook* or *rim hook* are also used. The height $ht(\nu/\lambda)$ of the border strip is the integer one less than the number of rows.

**Theorem 47.** *The product $s_\lambda p_n$ is equal to*

$$s_\lambda p_n = \sum_\nu (-1)^{ht(\nu/\lambda)} s_\nu,$$

*where all partitions $\lambda \subseteq \nu$ for which $\nu/\lambda$ is a border strip with n boxes.*

On the representation theory side, Murnaghan-Nakayama rule provides us with a recursive rule to calculate the values of irreducible characters of the symmetric group.

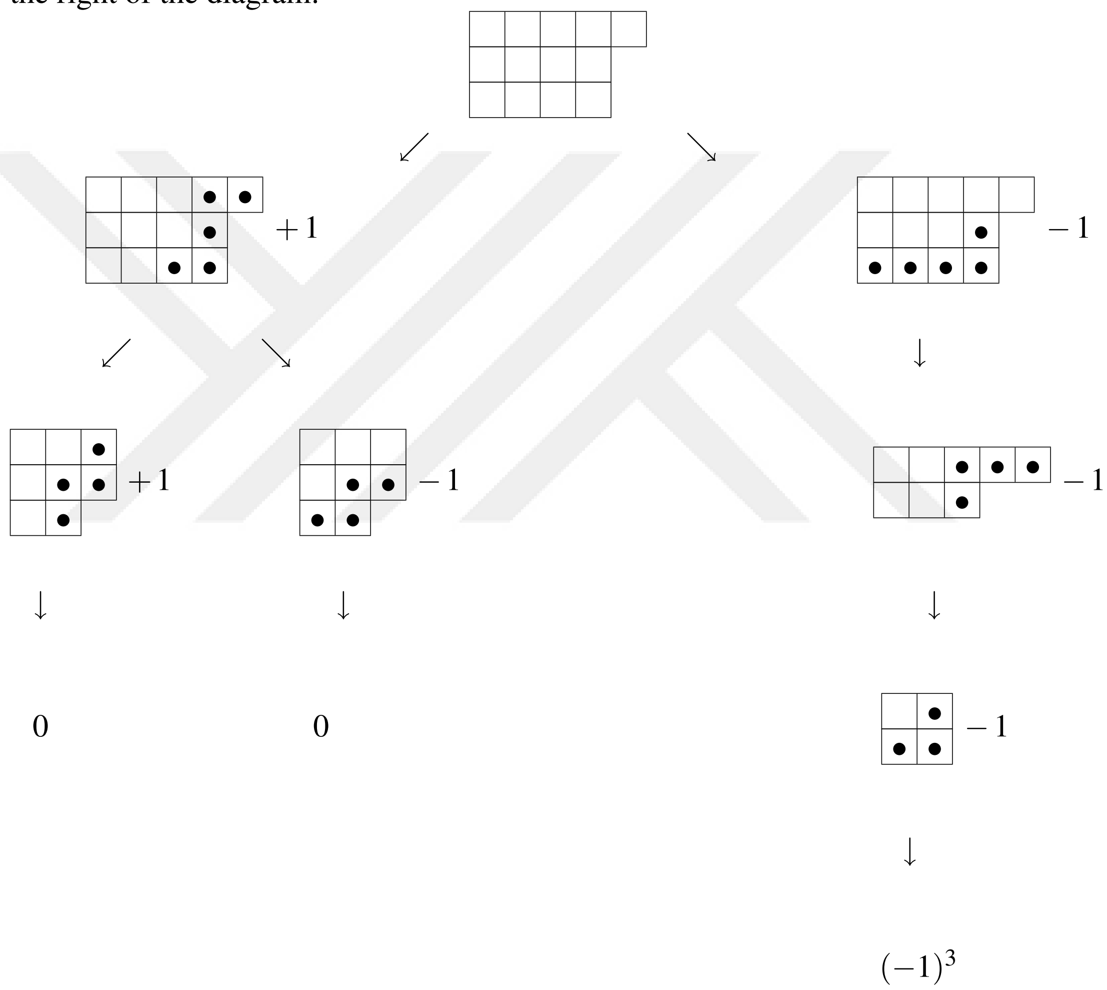
If $\nu/\lambda = \mu$ is a border strip, then we write $\nu\backslash\mu$ for $\lambda$. Moreover, if $\alpha = (\alpha_1, \alpha_2, ..., \alpha_r)$ is a composition, then by $\alpha\backslash\alpha_1$, denote $(\alpha_2, ..., \alpha_r)$.

**Theorem 48.** *If $\nu$ is a partition of $n$ and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_r)$ is a composition of $n$, then*

$$\chi^{\nu}_{\alpha} = \sum_{\mu} (-1)^{ht(\mu)}\, \chi^{\nu\backslash\mu}_{\alpha\backslash\alpha_1},$$

*summed over all border strips $\mu$ of $\nu$ so that $|\mu| = \alpha_1$.*

**Example 49.** We compute $\chi^{(5,4,4)}_{(5,4,3,1)}$. Stripping the hooks can be visualized as a tree, whose nodes are diagrams and whose arrows correspond to removing certain border strips. Boxes to be removed are marked with dots and the appropriate sign appears to the right of the diagram:




Therefore, by Theorem 48, we calculate

$$\begin{aligned}
\chi^{(5,4,4)}_{(5,4,3,1)} &= \chi^{(3,3,2)}_{(4,3,1)} - \chi^{(5,3)}_{(4,3,1)} \\
&= \big(\chi^{(2,1,1)}_{(3,1)} - \chi^{(3,1)}_{(3,1)}\big) - \big(-\chi^{(2,2)}_{(3,1)}\big) \\
&= (0+0) - \big(-\big(-\chi^{(1)}_{(1)}\big)\big) \\
&= (-1)^3 \\
&= -1.
\end{aligned}$$

#### 1.3.11.1 A Special Case

If $\lambda = \emptyset$, then $s_\lambda = 1$ and the Murnaghan-Nakayama rule gives the decomposition of the power symmetric function $p_n$ in the Schur basis:

$$p_n = \sum_{\nu} (-1)^{ht(\nu)} s_\nu,$$

where the sum runs over all border strips $\nu$ with $n$ boxes.

### 1.3.12 Macdonald Polynomials

The Macdonald polynomials $P_\lambda(x;q,t)$ are $q;t$-generalizations of the Schur functions and monomial symmetric functions. They span the ring

$$\Lambda_{\mathbb{F}} := \mathbb{F}[x_1,...,x_n]^{S_n}$$

where $\mathbb{F} = \mathbb{Q}(q,t)$.

For $t = q$, Macdonald polynomials specialize to the Schur functions:

$$P_\lambda(x;q,q) = s_\lambda(x).$$

For $t = 1$, Macdonald polynomials specialize to monomial symmetric functions:

$$P_\lambda(x;q,1) = m_\lambda(x).$$

Macdonald defined a $q;t$-analogue of Hall's scalar product by defining

$$\langle p_\lambda, p_\mu \rangle = \langle p_\lambda, p_\mu \rangle_{q,t} = \delta_{\lambda\mu} z_\lambda \prod_{i=1}^{l(\lambda)} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}. \tag{1.2}$$

In addition to Schur functions and monomial symmetric functions, Macdonald polynomials specialize to Hall-Littlewood and Jack polynomials under different limits.

For $q = 0$, Macdonald polynomials specialize to Hall-Littlewood functions $P_\lambda(x;t)$. To obtain Jack symmetric functions from Macdonald polynomials, set $q = t^\alpha$ for a positive real number $\alpha$ and let $t \to 1$, so that $(1-q^r)/(1-t^r) \to \alpha$ for each $r \geq 1$. The limit of the scalar product (1.2) is

$$\langle p_\lambda, p_\mu \rangle = z_\lambda \alpha^{\ell(\lambda)} \delta_{\lambda\mu}. \tag{1.3}$$

where $\delta_{\lambda\mu} = 0$ if $\lambda \neq \mu$ and $\delta_{\lambda\lambda} = 1$.

We are going to see (Equation 1.3) in next section.

### 1.3.13 Jack Polynomials

Denote the ring of symmetric functions over the field $\mathbb{Q}(\alpha)$ by $\Lambda(\alpha)$. Note that $\Lambda(\alpha) = \Lambda \otimes_{\mathbb{Q}} \mathbb{Q}(\alpha)$. The ring of symmetric functions $\Lambda(\alpha)$ comes equipped with the inner product $\langle \cdot, \cdot \rangle : \Lambda(\alpha) \otimes_{\mathbb{Q}(\alpha)} \Lambda(\alpha) \to \mathbb{Q}(\alpha)$ defined by Equation 1.3.

**Theorem 50.** *The following conditions uniquely determine the symmetric functions $J_\lambda = J_\lambda(x, \alpha) \in \Lambda(\alpha)$, where $\lambda$ ranges over all partitions:*

1. *(Orthogonality) $\langle J_\lambda, J_\mu \rangle = 0$ if $\lambda \neq \mu$.*

2. *(Triangularity) Suppose $J_\lambda = \sum_\mu v_{\lambda\mu} m_\mu$. Then $v_{\lambda\mu} = 0$ unless $\mu \leq \lambda$ .*

3. *(Normalization) If $|\lambda| = n$, then the coefficient $v_{\lambda,1^n}$ of $x_1 x_2 \ldots x_n$ in $J_\lambda$ is equal to $n!$.*

The resulting functions $J_\lambda(x; \alpha)$ are called *Jack polynomials*. They form a basis of symmetric functions $\Lambda(\alpha)$ over the field $\mathbb{Q}(\alpha)$.

If we set all but finitely many variables equal to 0 (say $x_{n+1} = x_{n+2} = \ldots = 0$) in $J_\lambda$, then we obtain a polynomial $J_\lambda(x_1, \ldots, x_n; \alpha)$ with coefficients in $\mathbb{Q}(\alpha)$.

We identify a partition $\lambda$ with its Young diagram $\{(i, j) : 1 \leq j \leq \lambda_i\}$.

**Definition 51.** For a box $x = (i, j) \in \lambda$, define the *hook length $h(x)$ of $\lambda$ at $x$* by

$$h(x) = h(i, j) = \lambda_i + \lambda'_j - i - j + 1.$$

Equivalently, $h(x)$ is the number of boxes directly to the right or directly below $x$, counting $x$ itself once.

**Example 52.** For the partition $\lambda = (4, 2, 1)$, here is the corresponding Young diagram with its each box filled with the corresponding hook length:

| | | | |
|---|---|---|---|
| 6 | 4 | 2 | 1 |
| 3 | 1 | | |
| 1 | | | |

.

Set

$$H_\lambda = \prod_{x \in \lambda} h(x),$$

the product of all hook-lengths of $\lambda$.

Jack functions $J_\lambda$ specialize to Schur functions $s_\lambda$ and Zonal polynomials $Z_\lambda$ in [3].

- $J_\lambda(x;1) = H_\lambda s_\lambda$ for $\alpha = 1$,

- $J_\lambda(x;2) = Z_\lambda$ for $\alpha = 2$.

### 1.3.14 Pieri Rule for Jack Functions

We define two $\alpha$-refinements of the hook-length $h_\lambda(i,j)$.

**Definition 53.** For $(i,j) \in \lambda$, respectively the *upper hook-length* $h^*(i,j)$ at $(i,j)$ and the *lower hook-length* $h_*(i,j)$ at $(i,j)$ are defined by the equations

$$h_\lambda^*(i,j) = h^*(i,j) = \lambda_j' - i + \alpha(\lambda_i - j + 1),$$

$$h_*^\lambda(i,j) = h_*(i,j) = \lambda_j' - i + 1 + \alpha(\lambda_i - j).$$

**Definition 54.** The set $A_\lambda(x)$ of squares directly to the right of $x = (i,j) \in \lambda$ is called the *arm* of $x$, of size $a_\lambda(x) = \# A_\lambda(x) = \lambda_i - j$.

**Definition 55.** The set $L_\lambda(x)$ of squares directly below $x \in \lambda$ is called the *leg* of $x$, of size $l_\lambda(x) = \# L_\lambda(x) = \lambda_j' - i$.

In terms of arms and legs, upper and lower hook-length can be calculated as

$$h_\lambda^*(x) = \alpha(a_\lambda(x) + 1) + l_\lambda(x),$$

$$h_*^\lambda(x) = \alpha a_\lambda(x) + l_\lambda(x) + 1.$$

**Proposition 56.** *[4] Let $n \geq 0$ and $(n) = (n,0,0,\ldots)$ be a partition. Then*

$$J_n = J_{(n)} = \sum_{\lambda \vdash n} \alpha^{n-l(\lambda)} n! z_\lambda^{-1} p_\lambda.$$

**Theorem 57.** $\langle J_\lambda, J_\lambda \rangle = \prod_{x\in\lambda} h_*^\lambda(x) h_\lambda^*(x)$.

**Proposition 58.** *[4] $\langle J_\mu J_n, J_\lambda \rangle \neq 0$ if and only if $\mu \subseteq \lambda$ and $\lambda/\mu$ is a horizontal $n$-strip.*

**Theorem 59.** *[4] Let $\mu \subseteq \lambda$, and let $\lambda/\mu$ be a horizontal $n$-strip. Then*

$$\langle J_\mu J_n, J_\lambda \rangle = \Big( \prod_{x\in\mu} A_{\lambda\mu}(x) \Big) \Big( \prod_{x\in(n)} h_n^*(x) \Big) \Big( \prod_{x\in\lambda} B_{\lambda\mu}(x) \Big),$$

*where $\prod_{x\in(n)} h_n^*(x) = n!\alpha^n$,*

$$A_{\lambda\mu}(x) = \begin{cases} h_*^\mu(x), & \text{if } \lambda/\mu \text{ does not contain a square in the same column as } x \\ h_\mu^*(x), & \text{otherwise} \end{cases}$$

*and*

$$B_{\lambda\mu}(x) = \begin{cases} h^*_\lambda(x), & \text{if } \lambda/\mu \text{ does not contain a square in the same column as } x \\ \\ h^\lambda_*(x), & \text{otherwise} \end{cases}.$$

The coefficient $\langle J_\mu J_n, J_\lambda \rangle$ is the coefficient of $J_\lambda$ in $J_\mu J_n$.

Pieri rule for Jack functions were established by Stanley [12] and for Macdonald polynomials by Macdonald [7].

**Claim 60.** *Sakamoto algorithm [11] in the following calculates the product of power sum symmetric functions and Jack symmetric functions.*

**Algorithm 61.** *[11] Take a general Young diagram $Y$ and $n > 0$. We parameterize $Y$ as $Y = Y^{(1)} = (s_1^{r_1-r_0}, s_2^{r_2-r_1}, \ldots, s_m^{r_m-r_{m-1}})$. Let $r_0 = 0$, $s_{m+1} = 0$. In this parameterization, the outer-corners of $Y^{(1)}$ are given by $(r_1, s_1), (r_2, s_2), \ldots, (r_m, s_m)$, where symbol "(1)" indicate that this diagram is going to be added by the first box.*

***Step 1:*** *Add one box, say to place $(r_{i_1-1}+1, s_{i_1}+1)$, to $Y^{(1)}$, and denote it as $Y^{(1)} \cup (r_{i_1-1}+1, s_{i_1}+1)$. Associate a coefficient to $Y^{(1)}$, which is given by*

$$\prod_{j=1}^{i_1-1} \frac{\left[\alpha\Big(a(r_j, s_{i_1}+1)+1\Big) + l(r_j, s_{i_1}+1)\right]}{\left[\alpha\Big(a(r_{j-1}+1, s_{i_1}+1)+1\Big) + \Big(l(r_{j-1}+1, s_{i_1}+1)+1\Big)\right]}$$

$$\times \prod_{j=i_1}^{m} \frac{\left[\alpha a(r_{i_1-1}+1, s_j) + \Big(l(r_{i_1-1}+1, s_j)+1\Big)\right]}{\left[\alpha\Big(a(r_{i_1-1}+1, s_{j+1}+1)+1\Big) + \Big(l(r_{i_1-1}+1, s_{j+1}+1)+1\Big)\right]}$$

$$= \prod_{j=1}^{i_1-1} \frac{h^*_Y(r_j, s_{i_1}+1)}{h^*_Y(r_{j-1}+1, s_{i_1}+1)+1} \times \prod_{j=i_1}^{m} \frac{h^Y_*(r_{i_1-1}+1, s_j)}{h^Y_*(r_{i_1-1}+1, s_{j+1}+1)+\alpha}$$

***Step 2:*** *Take a coordinate of $Y^{(2)} = Y^{(1)} \cup (r_{i_1-1}+1, s_{i_1}+1)$ as in $Y^{(1)}$, namely, set*

$$Y^{(2)} = Y^{(1)} \cup (r_{i_1-1}+1, s_{i_1}+1) = \left(s_1^{r_1-r_0}, s_2^{r_2-r_1}, \ldots, s_m^{r_m-r_{m-1}}\right).$$

*Add one more box to anywhere you want to this diagram if this addition gives us a Young diagram, and we obtain a similar factor as in Step 1. In this step, however, we need to work with the following "exception rule". If the outer-corner $(r_k, s_k)$ is produced by the box just added in the last step, we should omit the factor corresponding to $(r_k, s_k)$ in the numerator.*

***Step 3:*** *Denote the Young diagram after the second addition as* $Y^{(3)} = Y^{(2)} \cup (r_{i_2-1}+1, s_{i_2}+1)$. *Add third box to* $Y^{(3)}$ *and obtain similar factors as in Step* 2. *Note that we need to work with the exception rule. Repeat this manipulation recursively until n-th box is added.*

***Step 4:*** *Multiply*

$$(-1)^{\#\{k|r_{i_k-1}<r_{i_{k+1}-1}\}} J_{Y\cup(r_{i_1-1}+1,s_{i_1}+1)\cup\ldots\cup(r_{i_n-1}+1,s_{i_n}+1)},$$

*to the result of Step* 3.

***Step 5:*** *Repeat Steps* 1 *to* 4 *for each way to add* $n$ *boxes to* $Y$, *and sum up all the terms.*

*Formula for* $J_Y p_n$ *is obtained by doing Steps* 1 *to* 5.

## 2. THE PRODUCT OF JACK SYMMETRIC FUNCTIONS $J_\mu$ AND THE POWER SUM SYMMETRIC FUNCTION $p_1$ AND $p_2$ FOR ANY YOUNG DIAGRAM $\mu$

Our main tool will be Stanley's Pieri rule for Jack functions [4]. Here is the first step of our proof of Murnaghan-Nakayama rule for Jack polynomials.

**Theorem 62.** *For $k = 1$ Algorithm 61 for the product $J_\mu p_k$ in [11] holds.*

*Proof.* Given partitions $\mu$ and $\lambda$ so that $\lambda/\mu$ is the box $(r_{i_1-1}+1, s_{i_1}+1)$, Algorithm 61 calculates the coefficient of $J_\lambda$ in the product $J_\mu p_1$ as

$$\prod_{j=1}^{i_1-1} \frac{h^*_\mu(r_j, s_{i_1}+1)}{h^*_\mu(r_{j-1}+1, s_{i_1}+1)+1} \times \prod_{j=i_1}^{m} \frac{h_*^\mu(r_{i_1-1}+1, s_j)}{h_*^\mu(r_{i_1-1}+1, s_{j+1}+1)+\alpha}, \tag{2.1}$$

where $m$ is the number of outer boxes and $i_1$ is the number of place which can be added one box.

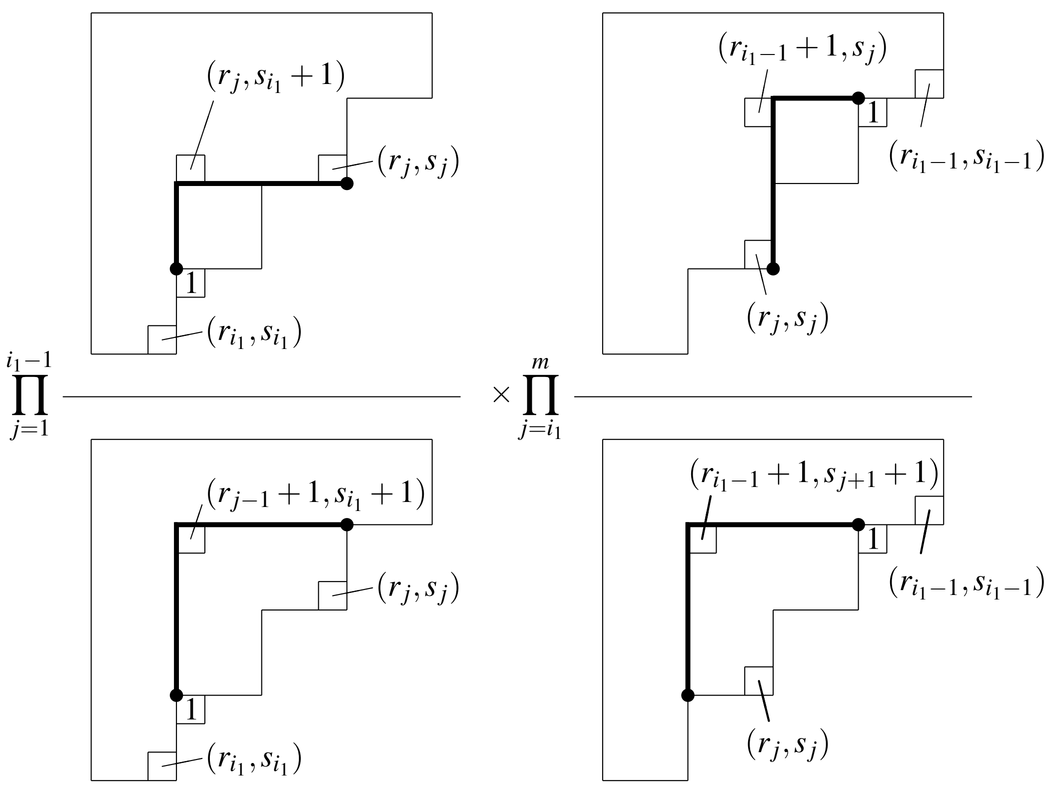



**Figure 2.1** : The coefficient of $J_\lambda$ in the product $J_\mu p_1$ where $\lambda/\mu$ is a single box

For a given diagram, define outer corners (black dots) and inner dots (white dots) as follows:

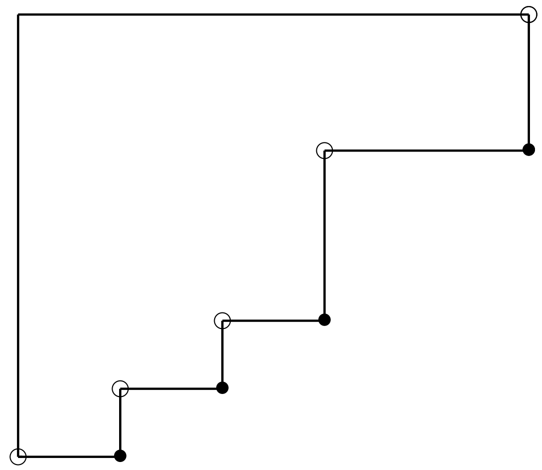


**Figure 2.2** : Outer and inner corners

Let partitions $\lambda$ and $\mu$ be so that $\lambda/\mu$ consists of a single box. Mark this box by "1" as in Figure 2.3. Furthermore, let $\eta$ denote boxes which are at the same column with box "1" and $\xi$ show boxes which are at the same row with box "1" as in the following figure:

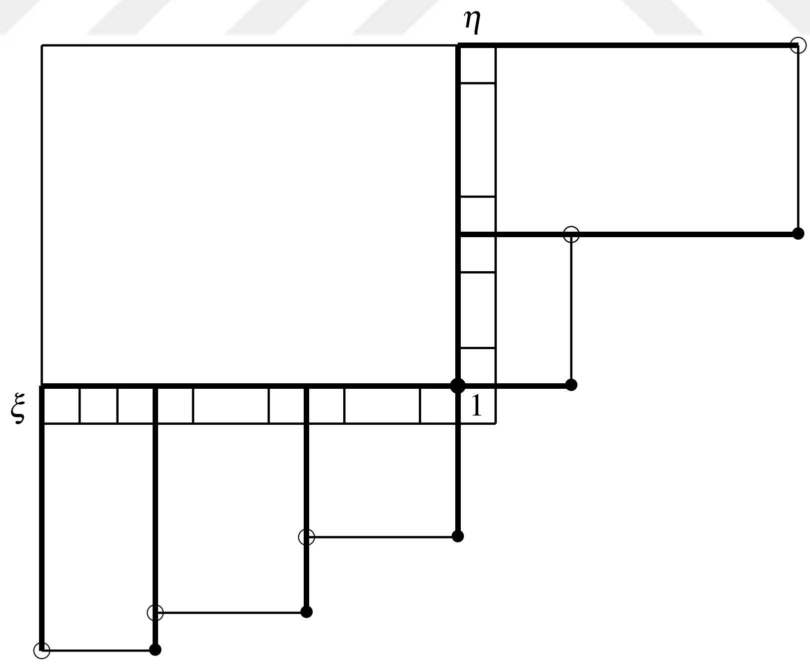



**Figure 2.3** : The product $J_\mu p_1$ where $\lambda/\mu$ consists of a single box

Using the Stanley's Pieri rule for Jack polynomials in Theorem 59 in Section (1.3.14), the coefficient of $J_\lambda$ in the product $J_\mu p_1$ as

$$
\begin{aligned}
[J_\lambda]J_\mu p_1 &= \frac{\left(\prod_{x\in\mu} A_{\lambda\mu}(x)\right)\left(\prod_{x\in\lambda} B_{\lambda\mu}(x)\right)\left(\prod_{x\in(n)} h_n^*(x)\right)}{j_\lambda} \\
&= \frac{\left(\prod_{x\in\mu\setminus\eta} h_*^\mu(x)\prod_{x\in\eta} h_\mu^*(x)\right)\left(\prod_{x\in\lambda\setminus(\eta\cup(\lambda\setminus\mu))} h_\lambda^*(x)\prod_{x\in\eta\cup(\lambda\setminus\mu)} h_*^\lambda(x)\right)\alpha}{\prod_{x\in\lambda} h_\lambda^*(x)\prod_{x\in\lambda} h_*^\lambda(x)} \\
&= \frac{\prod_{x\in\mu\setminus\eta} h_*^\mu(x)\prod_{x\in\eta} h_\mu^*(x)\ \ \alpha}{\prod_{x\in\eta\cup(\lambda\setminus\mu)} h_\lambda^*(x)\prod_{x\in\lambda\setminus(\eta\cup(\lambda\setminus\mu))} h_*^\lambda(x)} \\
&= \frac{\prod_{x\in\mu\setminus\eta} h_*^\mu(x)}{\prod_{x\in\lambda\setminus(\eta\cup(\lambda\setminus\mu))} h_*^\lambda(x)}\ \frac{\prod_{x\in\eta} h_\mu^*(x)}{\prod_{x\in\eta} h_\lambda^*(x)} \\
&= \prod_{x\in\mu\setminus\eta}\frac{h_*^\mu(x)}{h_*^\lambda(x)}\ \prod_{x\in\eta}\frac{h_\mu^*(x)}{h_\lambda^*(x)} \\
&= \prod_{x\in\xi}\frac{h_*^\mu(x)}{h_*^\lambda(x)}\ \prod_{x\in\eta}\frac{h_\mu^*(x)}{h_\lambda^*(x)}. \qquad (2.2)
\end{aligned}
$$

When $\lambda/\mu$ is a single box, say to place $(r_{i_1-1}+1, s_{i_1}+1)$ then using the Algorithm 61, we have the following formula:

$$
\begin{aligned}
\prod_{x\in\eta}\frac{h_\mu^*(x)}{h_\lambda^*(x)} &= \prod_{x\in\eta}\frac{h_\mu^*(x)}{h_\mu^*(x)+1} \\
&= \frac{h_\mu^*(r_1, s_{i_1}+1)}{(h_\mu^*(r_0+1, s_{i_1}+1)+1)}\ \frac{h_\mu^*(r_2, s_{i_1}+1)}{(h_\mu^*(r_1+1, s_{i_1}+1)+1)} \\
&\qquad \cdots \frac{h_\mu^*(r_{i_1-1}, s_{i_1}+1)}{(h_\mu^*(r_{i_1-2}+1, s_{i_1}+1))+1} \\
&= \prod_{j=1}^{i_1-1}\frac{h_\mu^*(r_j, s_{i_1}+1)}{h_\mu^*(r_{j-1}+1, s_{i_1}+1)+1}. \qquad (2.3)
\end{aligned}
$$

and

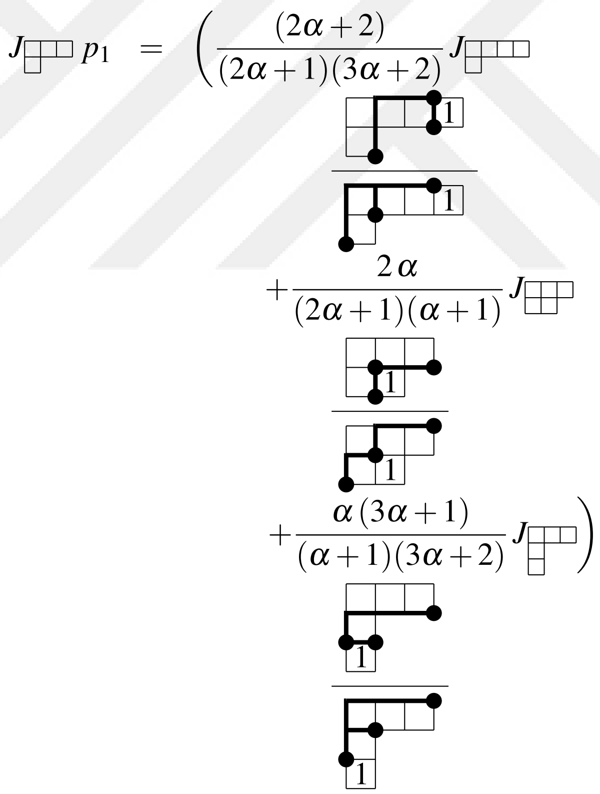
$$
\begin{aligned}
\prod_{x\in\xi}\frac{h_*^\mu(x)}{h_*^\lambda(x)} &= \prod_{x\in\xi}\frac{h_*^\mu(x)}{h_*^\mu(x)+\alpha} = \frac{h_*^\mu(r_{i_1-1}+1, s_{i_1})}{(h_*^\mu(r_{i_1-1}+1, s_{i_1+1}+1)+\alpha)}\frac{h_*^\mu(r_{i_1-1}+1, s_{i_1+1})}{(h_*^\mu(r_{i_1-1}+1, s_{i_1+2}+1)+\alpha)} \\
&\qquad \cdots \frac{h_*^\mu(r_{i_1-1}+1, s_m)}{(h_*^\mu(r_{i_1-1}+1, s_{m+1}+1)+\alpha)} \\
&= \prod_{j=1}^{m}\frac{h_*^\mu(r_{i_1-1}+1, s_j)}{(h_*^\mu(r_{i_1-1}+1, s_{j+1}+1)+\alpha)}. \qquad (2.4)
\end{aligned}
$$

Thus using the Equation 2.3 and Equation 2.4 we have the following formula;

$$
\begin{aligned}
[J_\lambda]J_\mu p_1 &= \prod_{x\in\eta}\frac{h^*_\mu(x)}{h^*_\lambda(x)}\prod_{x\in\xi}\frac{h^\mu_*(x)}{h^\lambda_*(x)} \\
&= \prod_{j=1}^{i_1-1}\frac{h^*_\mu(r_j,s_{i_1}+1)}{h^*_\mu(r_{j-1}+1,s_{i_1}+1)+1}\prod_{j=i_1}^{m}\frac{h^\mu_*(r_{i_1-1}+1,s_j)}{h^\mu_*(r_{i_1-1}+1,s_{j+1}+1)+\alpha}. \qquad (2.5)
\end{aligned}
$$

We show that the Equation 2.2 is equal to the Equation 2.5. Hence we prove that Algorithm 61 and Stanley's Pieri rule for Jack polynomials coincide whenever $\lambda/\mu$ is a single box. $\square$

As an example in [11], let's calculate $J_{(3,1)}p_1$:



## 2.1 The Relation of Jack Symmetric Functions $J_n$ and the Power Sum Symmetric Functions $p_n$ for any Positive Number $n$

**Definition 63.** [2] Given a sequence $(a_n)_{n\geq 0} = a_0, a_1, a_2, ...$ of complex numbers, the corresponding *generating function* is the power series

$$f(x) = \sum_{n\geq 0} a_n x^n.$$

If numbers $a_n$ enumerate some set of combinatorial objects, then we say that $f(x)$ is the generating function for those objects. We also write

$$[x^n]f(x) = \text{the coefficient of } x^n \text{ in } f(x) = a_n.$$

Also the generating function for the $r$-th complete symmetric function $h_r$ is in [3]

$$H(t) = \sum_{r\geq 0} h_r t^r = \prod_{i\geq 1}(1-x_i t)^{-1}.$$

The generating function for the $r$-th power sum is in [3]

$$P(t) = \sum_{r\geq 1} p_r t^{r-1} = \sum_{i\geq 1}\sum_{r\geq 1} x_i^r t^{r-1} = \sum_{i\geq 1} \frac{x_i}{1-x_i t} = \sum_{i\geq 1} \frac{d}{dt} log \frac{1}{1-x_i t}$$

so that

$$P(t) = \frac{d}{dt} log \prod_{i\geq 1}(1-x_i t)^{-1} = \frac{d}{dt} log H(t) = \frac{H'(t)}{H(t)}.$$

Hence,

$$\begin{aligned} H'(t) &= P(t)H(t) \\ nh_n t^{n-1} &= (p_1 + p_2 t + p_3 t^2 + ...)(1 + h_1 t + ...) \\ nh_n t^{n-1} &= (p_1 h_{n-1} + ... + p_{n-1} h_1 + p_n) t^{n-1} \\ nh_n &= p_1 h_{n-1} + ... + p_{n-1} h_1 + p_n. \end{aligned} \tag{2.6}$$

A similar identity is satisfied by power sum symmetric functions $p_k$ and Jack polynomials $J_k$ as well:

Using the formulas,

$$J_n = \sum_{\lambda \vdash n} \alpha^{n-\ell(\lambda)} n! z_\lambda^{-1} p_\lambda$$

and

$$J(t)=\sum_{n\geq 0} J_n(x)\frac{t^n}{\alpha^n n!}=\prod_i (1-x_i t)^{-\frac{1}{\alpha}},$$

in [4] we have:

$$\begin{aligned}
logJ(t) &= \frac{1}{\alpha}\sum log\left(\frac{1}{1-x_i t}\right)\\
logJ(t) &= \frac{1}{\alpha} logH(t)\\
J^{\alpha}(t) &= H(t).
\end{aligned}$$

Then,

$$\begin{aligned}
H'(t)=P(t)H(t) &\Rightarrow \alpha J^{\alpha-1}(t)J'(t)=P(t)J^{\alpha}(t)\\
&\Rightarrow \alpha J'(t)=J(t)P(t). \qquad (2.7)
\end{aligned}$$

Therefore,

$$J(t)=\sum_{n\geq 0} J_n(x)\frac{t^n}{n!\alpha^n}\Rightarrow J'(t)=\sum_{n\geq 1} J_n(x)\frac{nt^{n-1}}{n!\alpha^n}.$$

By Equation 2.7,

$$\begin{aligned}
\alpha J'(t) &= J(t)P(t)\\
\alpha\left(\sum_{n\geq 1} J_n(x)\frac{t^{n-1}}{(n-1)!\alpha^n}\right) &= \sum_{n\geq 0} J_n(x)\frac{t^n}{n!\alpha^n}(p_1+p_2t+p_3t^2+...)\\
\sum_{n\geq 1} J_n\frac{t^{n-1}}{(n-1)!\alpha^{n-1}} &= \left(1+J_1\frac{t}{\alpha}+J_2\frac{t^2}{2!\alpha^2}+J_3\frac{t^3}{3!\alpha^3}+...\right)(p_1+p_2t+p_3t^2+...)\\
&= p_1+p_2t+p_3t^2+...\\
&\qquad +J_1p_1\frac{t}{\alpha}+J_1p_2\frac{t^2}{\alpha}+J_1p_3\frac{t^3}{\alpha}+...\\
&\qquad +J_2p_1\frac{t^2}{2!\alpha^2}+J_2p_2\frac{t^3}{2!\alpha^2}+J_2p_3\frac{t^4}{2!\alpha^2}+...\\
&\qquad +J_3p_1\frac{t^3}{3!\alpha^2}+J_3p_2\frac{t^4}{3!\alpha^3}+J_3p_3\frac{t^5}{3!\alpha^3}+...\\
&= p_1+\left(p_2+\frac{J_1p_1}{\alpha}\right)t+\left(p_3+\frac{J_1p_2}{\alpha}+\frac{J_2p_1}{2\alpha^2}\right)t^2\\
&\qquad +\left(p_4+\frac{J_1p_3}{\alpha}+\frac{J_2p_2}{2!\alpha^2}+\frac{J_3p_1}{3!\alpha^2}\right)t^3+...\\
&= \sum_{r=1}^{n}\frac{J_{n-r}p_r}{(n-r)!\alpha^{n-r}}t^{n-1}.
\end{aligned}$$

Hence,

$$\frac{J_n}{(n-1)!\alpha^{n-1}} = \sum_{r=1}^{n} \frac{J_{n-r}p_r}{(n-r)!\alpha^{n-r}}$$

$$J_n = \sum_{r=1}^{n} \frac{J_{n-r}\, p_r\, (n-1)!\, \alpha^{r-1}}{(n-r)!}. \tag{2.8}$$

## 2.2 The Product of Jack Polynomials $J_\mu$ and $J_n$ for any Young diagram $\mu$

Let partitions $\lambda$ and $\mu$ be so that $\lambda/\mu$ consists of $\lambda_1/\mu_1 = k_1, \lambda_2/\mu_2 = k_2, ..., \lambda_m/\mu_m = k_m$ boxes. Let $\eta$'s denote boxes which are on the same column with the adding boxes and $\xi$'s denote boxes which are on the same row with the adding boxes as in the following figure:

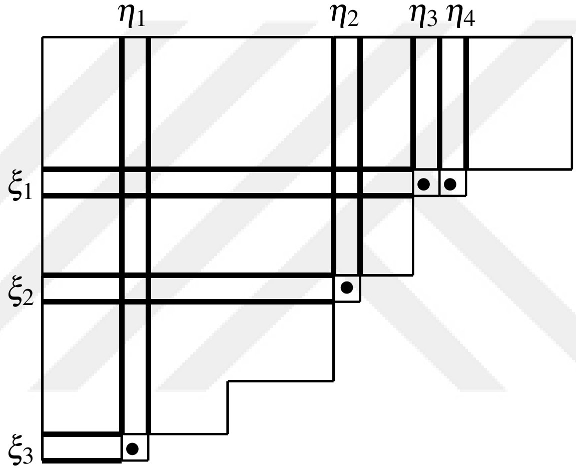



**Figure 2.4** : The product of $J_\mu$ and $J_n$ for any Young diagram $\mu$

Then the coefficient of $J_\lambda$ in the product $J_\mu J_n$ where $\lambda/\mu = (k_1, ..., k_m)$ is

$$\begin{aligned}
[J_\lambda]J_\mu J_n &= \prod_{x\in\xi\backslash\eta} \frac{h_*^\mu(x)}{h_*^\lambda(x)} \prod_{x\in\eta} \frac{h_\mu^*(x)}{h_\lambda^*(x)} \frac{n!\alpha^n}{\prod_{\lambda/\mu} h_\lambda^*(x)} \\
&= \prod_{x\in\xi\backslash\eta} \frac{h_*^\mu(x)}{h_*^\lambda(x)} \prod_{x\in\eta} \frac{h_\mu^*(x)}{h_\lambda^*(x)} \binom{n}{k_1...k_m}.
\end{aligned} \tag{2.9}$$

## 2.3 The Relation of the Product $J_\mu\ p_2$ and the Product $J_\mu\ p_1^2$

**Theorem 64.** *Algorithm 61 holds for $J_\mu p_2$. The coefficient of $J_\lambda$ such that $\lambda = (\mu, 1, 1)$ in $J_\mu p_2$ in Algorithm 61 is the coefficient of $-\frac{1}{\alpha} J_\mu\ {p_1}^2$ in Theorem 59.*

*Proof.* For starters, calculate $p_2$ in terms of Jack polynomials:

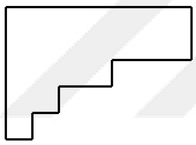
$$\begin{aligned} J_1 &= p_1, \\ J_2 &= \alpha p_2 + p_1\, J_1 = \alpha p_2 + {p_1}^2, \\ p_2 &= \frac{1}{\alpha}(J_2 - {p_1}^2) = \frac{1}{\alpha}(J_2 - {J_1}^2). \end{aligned}$$

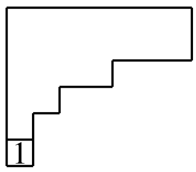
Hence,

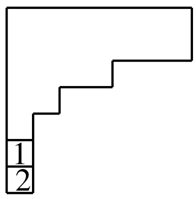
$$\begin{aligned} J_\mu\, p_2 &= \frac{1}{\alpha}(J_\mu\, J_2 - J_\mu\, {J_1}^2). \\ J_\mu\, p_2 &= \frac{1}{\alpha}(J_\mu\, J_2 - J_\mu\, {p_1}^2). \end{aligned} \tag{2.10}$$

Firstly, using Proposition 58 we know that $\langle J_\mu J_2, J_\lambda \rangle = 0$ is in the Equation 2.10 for $\lambda = (\mu, 1, 1)$.

We determine the relation between the coefficient of $J_{\lambda=(\mu,1,1)}$ in the Theorem 59 and the coefficient of $J_{\lambda=(\mu,1,1)}$ in the Algorithm 61.

Let $\mu =$ 

Then $\nu = (\mu, 1) =$ 


and $\lambda = (\mu, 1, 1) =$ 


Firstly,

$$J_\mu\, p_1 = \prod_{x\in\eta} \frac{h^*_\mu(x)}{h^*_\nu(x)} \prod_{x\in\mu\setminus\eta} \frac{h^\mu_*(x)}{h^\nu_*(x)} J_\nu = \prod_{x\in\eta} \frac{h^*_\mu(x)}{h^*_\nu(x)} J_\nu \quad \Rightarrow \quad J_\mu\, p_1^2 = \prod_{x\in\eta} \frac{h^*_\mu(x)}{h^*_\nu(x)} J_\nu\, p_1$$

$$
\begin{aligned}
J_\nu\, p_1 &= \prod_{x\in\eta\cup\nu\setminus\mu} \frac{h^*_\nu(x)}{h^*_\lambda(x)} \prod_{x\in\nu\setminus(\eta\cup\nu\setminus\mu)} \frac{h^\nu_*(x)}{h^\lambda_*(x)}\, J_\lambda \\
&= \prod_{x\in\eta} \frac{h^*_\nu(x)}{h^*_\lambda(x)}\, \frac{h^*_\nu(\nu\setminus\mu)}{h^*_\lambda(\nu\setminus\mu)}\, J_\lambda \\
&= \prod_{x\in\eta} \frac{h^*_\nu(x)}{h^*_\lambda(x)}\, \frac{\alpha}{\alpha+1}\, J_\lambda
\end{aligned}
$$

Hence

$$
\begin{aligned}
J_\mu\, p_1^2 &= \prod_{x\in\eta} \frac{h^*_\mu(x)}{h^*_\nu(x)} J_\nu\, p_1 \\
&= \prod_{x\in\eta} \frac{h^*_\mu(x)}{h^*_\nu(x)}\, \frac{h^*_\nu(x)}{h^*_\lambda(x)}\, \frac{\alpha}{\alpha+1}\, J_\lambda
\end{aligned}
$$

$$
\begin{aligned}
[J_\lambda]J_\mu p_2 = -\frac{1}{\alpha} J_\mu\, p_1^2 &= -\prod_{x\in\eta} \frac{h^*_\mu(x)}{h^*_\lambda(x)(\alpha+1)}\, J_\lambda \\
&= -\prod_{x\in\eta} \frac{h^*_\mu(x)}{(h^*_\mu(x)+2)(\alpha+1)}\, J_\lambda \\
&= -\frac{1}{(\alpha+1)} \prod_{x\in\eta} \frac{h^*_\mu(x)}{(h^*_\mu(x)+2)}\, J_\lambda \qquad (2.11)
\end{aligned}
$$

Secondly, by the Algorithm 61 we have,

$$
\begin{aligned}
[J_\lambda]J_\mu p_2 &= \prod_{j=1}^{i_1-1} \frac{h^*_\mu(r_j, s_{i_1}+1)}{h^*_\mu(r_{j-1}+1, s_{i_1}+1)+1} \prod_{j=1}^{i_2-1} \frac{h^*_\nu(r_j, s_{i_1}+1)}{h^*_\nu(r_{j-1}+1, s_{i_1}+1)+1} \\
&= \frac{-1}{\alpha+1} \prod_{j=1}^{i_1-1} \frac{h^*_\mu(r_j, s_{i_1}+1)}{h^*_\mu(r_{j-1}+1, s_{i_1}+1)+1} \prod_{j=1}^{i_2-1} \frac{h^*_\mu(r_j, s_{i_1}+1)+1}{h^*_\mu(r_{j-1}+1, s_{i_1}+1)+2} \\
&= -\frac{1}{\alpha+1} \prod_{k=1}^{2} \prod_{j=1}^{i_1-1} \frac{h^*_\mu(r_j, s_{i_k}+1)+(k-1)}{h^*_\mu(r_{j-1}+1, s_{i_k}+1)+k} \qquad (2.12)
\end{aligned}
$$

Hence the factors in the Equation 2.11 and the factors in the Equation 2.12 coincide.

□

## 2.4 Some Computations

We calculate some products for some specific Young diagrams. Using them we obtain some general formulas.

Let $n$ be a nonnegative integer. Given the partition $\mu = n$ so that $\lambda/\mu$ is a single box, we have the following formula:

$$J_n p_1 = \frac{\alpha n}{\alpha n+1} J_{(n,1)} + \frac{1}{\alpha n+1} J_{(n+1)},$$

Moreover, let the partition $\mu = 1^n$ and $\lambda/\mu$ is the one box. We obtain the following formula:

$$J_{1^n} p_1 = \frac{n}{\alpha+n} J_{(2,1^{n-1})} + \frac{\alpha}{\alpha+n} J_{1^{n+1}}.$$

Let the partition $\mu = n$. When $\lambda/\mu$ equals to the two, three and four boxes respectively, we have the followings:

$$\begin{aligned}
J_n p_2 &= \frac{1}{\Big((n+1)\alpha+1\Big)(\alpha n+1)} J_{(n+2)} \\
&\quad + \frac{2(\alpha-1)}{(\alpha n+2)\Big((n+1)\alpha+1\Big)\Big((n-1)\alpha+1\Big)} J_{(n+1,1)} \\
&\quad + \frac{(n-1)\alpha^2 n}{(\alpha n+1)(\alpha+1)\Big((n-1)\alpha+1\Big)} J_{(n,2)} \\
&\quad - \frac{\alpha n}{(\alpha n+2)(\alpha n+2)(\alpha+1)} J_{(n,1,1)},
\end{aligned}$$

$$\begin{aligned}
J_n p_3 &= \frac{1}{(\alpha n+1)(\alpha(n+1)+1)(\alpha(n+2)+1)} J_{(n+3)} \\
&\quad + \frac{3(\alpha-1)}{(\alpha(n+2)+1)(\alpha(n-1)+1)(\alpha(n+1)+2)(\alpha n+1)} J_{(n+2,1)} \\
&\quad + \frac{3\alpha n(\alpha-1)}{(\alpha n+1)(\alpha+1)(\alpha(n+1))(\alpha n+2)(\alpha(n-2)+1)} J_{(n+1,2)} \\
&\quad - \frac{3(\alpha-1)}{(\alpha+1)(\alpha(n+1)+2)(\alpha(n-1)+1)(\alpha n+3)} J_{(n+1,1,1)} \\
&\quad + \frac{\alpha^3 n(n-1)(n-2)}{(\alpha n+1)(\alpha(n-1)+1)(\alpha+1)(\alpha(n-2)+1)(2\alpha+1)} J_{(n,3)} \\
&\quad + \frac{3\alpha^2 n(n-1)}{(\alpha(n-1)+1)(\alpha+2)(\alpha n+2)(2\alpha+1)} J_{(n,2,1)} \\
&\quad + \frac{\alpha n}{(\alpha+1)(\alpha+2)(\alpha n+3)} J_{(n,1,1,1)},
\end{aligned}$$

$$
\begin{aligned}
J_n p_4 \quad = \quad & \frac{1}{(\alpha n+1)(\alpha(n+1)+1)(\alpha(n+2)+1)(\alpha(n+3)+1)} J_{(n+4)} \\
& + \frac{4(\alpha-1)}{(\alpha(n+2)+2)(\alpha(n-1)+1)(\alpha(n+1)+1)} \\
& \qquad \cdot \frac{1}{(\alpha(n+3)+1)(\alpha n+1)} J_{(n+3,1)} \\
& + \frac{2(\alpha-1)(2\alpha^2(n^2-1)+\alpha(4n+3)-1)}{(\alpha(n+1)+1)(\alpha(n+1)+2)(\alpha+1)(\alpha(n-1)+1)} \\
& \qquad \cdot \frac{1}{(\alpha n+2)(\alpha(n-2)+1)} J_{(n+2,2)} \\
& - \frac{4(\alpha-1)}{(\alpha n+1)(\alpha+1)(\alpha(n+2)+2)(\alpha(n-1)+1)} \\
& \qquad \cdot \frac{1}{(\alpha(n+1)+3)} J_{(n+2,1,1)} \\
& + \frac{4\alpha^2 n(n-1)}{(\alpha n+1)(\alpha(n+1)+1)(\alpha n+2)(\alpha(n-1)+1)(2\alpha+1)} \\
& \qquad \cdot \frac{1}{(\alpha+1)(\alpha(n-3)+1)} J_{(n+1,3)} \\
& - \frac{4(\alpha(3n-4)+7)(\alpha-1)\alpha n}{(\alpha n+1)(\alpha(n+1)+2)(\alpha(n-1)+2)(2\alpha+1)(\alpha+2)} \\
& \qquad \cdot \frac{1}{(\alpha(n-2)+1)(\alpha n+3)} J_{(n+1,2,1)} \\
& + \frac{4(\alpha-1)}{(\alpha(n-1)+1)(\alpha n+4)(\alpha+1)(\alpha+2)(\alpha(n+1)+3)} J_{(n+1,1,1,1)} \\
& + \frac{\alpha^4 n(n-1)(n-2)(n-3)}{(\alpha n+1)(\alpha+1)(2\alpha+1)(3\alpha+1)(\alpha(n-1)+1)(\alpha(n-2)+1)} \\
& \qquad \cdot \frac{1}{(\alpha(n-3)+1)} J_{(n,4)}
\end{aligned}
$$

$$-\frac{4\alpha^3 n(n-1)(n-2)}{(2\alpha+2)(3\alpha+1)(\alpha n+2)(\alpha+1)(\alpha(n-2)+1)}$$

$$\cdot\frac{1}{(\alpha(n-1)+1)}J_{(n,3,1)}$$

$$-\frac{\alpha^2 n(n-1)(\alpha-1)}{(2\alpha+1)(\alpha n+2)(\alpha+2)^2(\alpha(n-1)+2)}J_{(n,2,2)}$$

$$+\frac{4\alpha^2 n(n-1)}{(\alpha+1)(2\alpha+2)(\alpha+3)(\alpha n+3)(\alpha(n-1)+1)}J_{(n,2,1,1)}$$

$$-\frac{\alpha n}{(\alpha+1)(\alpha+2)(\alpha+3)(\alpha n+4)}J_{(n,1,1,1,1)}.$$

Using similar methods, we also computed $J_n\, p_5$ which has nineteen Young diagram which we do not include here.

Moreover, we give a general formula for positive integers $n$ and $k$ in the following:

$$\begin{aligned}
J_n p_k \quad = \quad & \frac{1}{(\alpha n+1)(\prod_{i=1}^{k-1}(\alpha(n+i)+1))}J_{(n+k)} \\
& + \frac{(-1)^{k-1}\alpha n}{(\alpha n+k)(\prod_{i=1}^{k-1}(\alpha+i))}J_{(n,1^k)} \\
& + \frac{\alpha^k \prod_{i=1}^{k}(n-(i-1))}{(\alpha(n-(i-1))+1)\prod_{i=1}^{k}((i-1)\alpha+1)}J_{(n,k)} \\
& + \frac{k(\alpha-1)}{(\alpha(n-1)+1)(\alpha(n+(k-2))+2)(\alpha(n+(k-1))+1)} \\
& \qquad \cdot\frac{1}{\prod_{i=1}^{k-2}(\alpha(n+(i-1))+1)}J_{(n+(k-1),1)} \\
& - \frac{k\alpha^{k-1}\prod_{i=1}^{k-1}(n-(i-1))}{(\alpha n+2)((k-2)\alpha+2)((k-1)\alpha+1)} \\
& \qquad \cdot\frac{1}{(\prod_{i=1}^{k-2}(\alpha(n-i)))(\prod_{i=1}^{k-3}(i\alpha+1))}J_{(n,k-1,1)}
\end{aligned}$$

$$+\frac{k\alpha^{k-2}(\alpha-1)\prod_{i=1}^{k-2}(n-(i-1))}{(\alpha n+1)(\alpha(n+1)+1)(\alpha n+2)(\alpha(n-(k-1))+1)}$$

$$\cdot\frac{1}{(\prod_{i=1}^{k-2}(i\alpha+1))(\prod_{i=1}^{k-3}(\alpha(n-i)+1))}J_{(n+1,k-1)}$$

$$+\frac{(-1)^{k}k(\alpha-1)}{(\alpha(n-1)+1)(\alpha(n+1)+(k-1))(\alpha n+k)}$$

$$\cdot\frac{1}{(\prod_{i=1}^{k-2}(\alpha+i))}J_{(n+1,1^{k-1})}$$

$$+\frac{(-1)^{k-1}k(\alpha-1)}{(\alpha n+1)(\alpha(n-1)+1)(\alpha(n+2)+(k-2))}$$

$$\cdot\frac{1}{(\alpha(n+1)+(k-1))}J_{(n+2,1^{k-2})}$$

$$+\frac{kn(n-1)\alpha^{2}}{(\alpha(n-1)+1)(\alpha n+(k-1))(2\alpha+(k-2))(\alpha+(k-1))}$$

$$\cdot\frac{1}{(\prod_{i=1}^{k-3}(\alpha+i))}J_{(n,2,1^{k-2})}$$

$$+\ldots$$

Let $\mu$ be an $m \times n$ rectangular diagram. The product of $J_\mu$ and $p_2$ where $\lambda/\mu$ consists of two boxes as in the following figure:

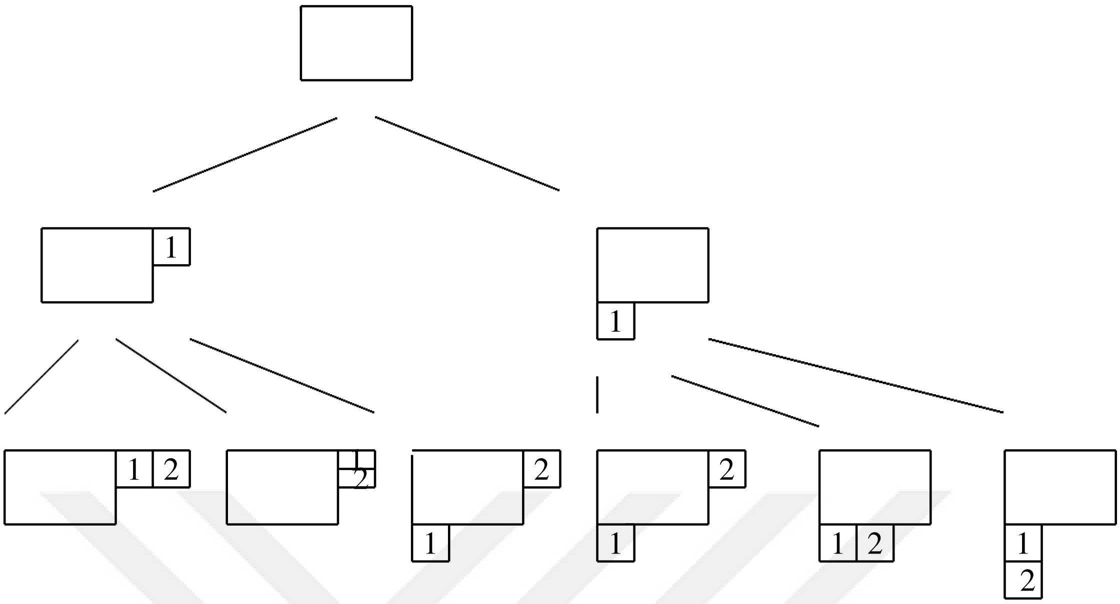
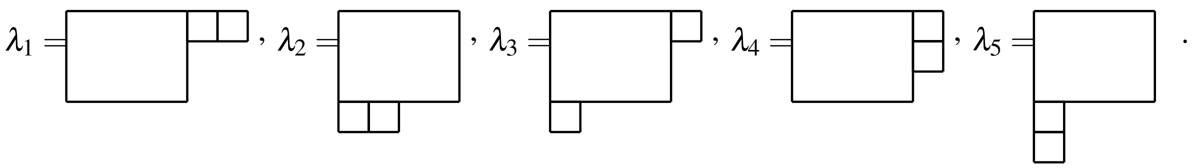


**Figure 2.5** : The product of $J_\mu$ and $p_2$ for an $m \times n$ rectangular diagram $\mu$

Hence we have the following formula:

$$\begin{aligned} J_\mu p_2 \quad = \quad & \frac{m(\alpha+m)}{\Big((n+1)\alpha+m\Big)(n\alpha+m)(\alpha+1)} J_{\lambda_1} + \frac{n(n-1)\alpha^2}{(n\alpha+m)\Big((n-1)\alpha+m\Big)(\alpha+1)} J_{\lambda_2} \\ & + \frac{2mn\alpha(\alpha-1)}{((n+1)\alpha+m)(n\alpha+(m-1))((n-1)\alpha+m)(n\alpha+(m-1))} J_{\lambda_3} \\ & - \frac{m(m-1)}{(n\alpha+m)\Big(n\alpha+(m-1)\Big)(\alpha+1)} J_{\lambda_4} - \frac{n\alpha(n\alpha+1)}{(n\alpha+m)\Big(n\alpha+(m+1)\Big)(\alpha+1)} J_{\lambda_5} \end{aligned}$$

where




Also we obtain the following corollary for the Young diagram $1^m$.

**Corollary 65.** *Let $n$ and $k$ be positive integers. When $m = n - k$;*

$$J_{1^m} J_k = \frac{m}{\alpha k + m} J_{(k+1, 1^{m-1})} + \frac{\alpha k}{\alpha k + m} J_{(k, 1^m)}.$$

### 2.5 The Product $J_\mu p_2$

We know the following equality by using Equation 2.10,

$$[J_\lambda]J_\mu p_2 = \frac{1}{\alpha}\Big[[J_\lambda]J_\mu J_2 - [J_\lambda]J_\mu J_1 J_1\Big].$$

Then

$$\begin{aligned}
[J_\lambda]J_\mu p_2 \quad = \quad & \prod_{j=i_1}^{m} \frac{h_*^\mu(r_{i_1-1}+1,s_j)}{h_*^\mu(r_{i_1-1}+1,s_{j+1}+1)+\alpha} \prod_{j=1}^{i_2-1} \frac{h_\mu^*(r_j,s_{i_2+1})}{h_\mu^*(r_{j-1}+1,s_{i_2}+1)+1} \\
& \prod_{j=i_2}^{i_1-2} \frac{h_*^\mu(r_{i_2-1}+1,s_j)}{h_*^\mu(r_{i_2-1}+1,s_{j+1}+1)+\alpha} \prod_{j=i_1-1}^{i_1-1} \frac{h_*^\mu(r_{i_2-1}+1,s_j)}{h_*^\mu(r_{i_2-1}+1,s_{j+1}+1+1)+\alpha} \\
& \prod_{j=i_1}^{m} \frac{h_*^\mu(r_{i_2-1}+1,s_j)}{h_*^\mu(r_{i_2-1}+1,s_{j+1}+1)+\alpha} \prod_{j=1}^{i_2-1} \frac{h_\mu^*(r_j,s_{i_1}+1)}{h_\mu^*(r_{j-1}+1,s_{i_1+1}+1)+1} \\
& \prod_{j=i_2}^{i_2} \frac{h_\mu^*(r_j,s_{i_1}+1)}{h_\mu^*(r_{j-1}+1+1,s_{i_1}+1)+1} \prod_{j=i_2+1}^{i_1-1} \frac{h_\mu^*(r_j,s_{i_1}+1)}{h_\mu^*(r_{j-1}+1,s_{i_1}+1)+1} \\
& \frac{2(\alpha-1)}{(h_*^\mu(r_{i_2-1}+1,s_{i_1}+1)+\alpha+1)(h_\mu^*(r_{i_2-1}+1,s_{i_1}+1)+\alpha+1)}.
\end{aligned}$$

Briefly, we obtained the following results:

1. An algorithm to calculate $J_\mu\ p_k$ is proposed by Sakamoto et al. in [11].

2. Computation verification has been done for partitions of small size. We computed the expansion of $J_\mu p_k$ which is claimed by Sakamoto et al. in [11] for the cases $\mu = (n)$, $\mu = (n,1)$ and $k \in \{1,2,3\}$ when $n \geq 1$.

3. $J_\mu\ p_1$ calculated by Sakamoto et al. holds. Hence we verified that the algorithm of $J_\mu p_k$ for $k = 1$ in [11] holds by comparing to the results in [4]. This is the first step of our main proof about Murnaghan-Nakayama rule for Jack polynomials.

4. We express $J_n$ with respect to Jack functions $J_0, J_1, ..., J_{n-1}$ and power symmetric functions $p_1, p_2, ..., p_n$ where $n \geq 1$.

5. The coefficient of $J_\lambda$ such that $\lambda = (\mu,1,1)$ in $J_\mu p_2$ is the coefficient of $-\frac{1}{\alpha}J_\mu {p_1}^2$. We showed that the coefficient of $J_\lambda$ is the same as the coefficient of $J_{\lambda=(\mu,1,1)}$ in the algorithm of Sakamoto et al..

6. We computed $J_{\mu} p_2$ where $\mu$ is the $m \times n$ rectangular diagram.

7. We compared the expansion of $J_{\mu} J_n$ proven by Stanley in [4] with the expansion of $J_{\mu} p_k$ indicated by Sakamoto et al. in [11] and we pointed out they are exactly equivalent to each other.

# 3. THE PRODUCT $J_\mu p_n$ FOR SOME CASES

We studied the products $J_\mu p_1$ and $J_\mu p_2$ when $\mu$ is an arbitrary Young diagram in the previous sections. We find the formulas of the product $J_\mu p_n$ for the following cases and hence the formula of the product $J_\mu p_n$ is generalized partially in this chapter.

## 3.1 The Coefficient of $J_\lambda$ in the Product $J_\mu p_n$ where $\lambda/\mu$ is a Column of $n$ Boxes

**Theorem 66.** *Let $\lambda$ and $\mu$ be two Young diagrams whose difference $\lambda/\mu$ is a single column of n boxes. Algorithm 61 for $J_\mu p_n$ calculates the coefficient of $J_\lambda$.*

*Proof.* By Equation 2.8,

$$\begin{aligned} J_n &= J_{n-1}p_1 + J_{n-2}p_2(n-1)\alpha + J_{n-3}p_3(n-1)(n-2)\alpha^2 \\ &\qquad + \ldots + J_1 p_{n-1}(n-1)!\alpha^{n-2} + p_n(n-1)!\alpha^{n-1}. \end{aligned}$$

Then,

$$p_n = \frac{J_n - J_{n-1}p_1 - J_{n-2}p_2(n-1)\alpha - J_{n-3}p_3(n-1)(n-2)\alpha^2 - \ldots - J_1 p_{n-1}(n-1)!\alpha^{n-2}}{(n-1)!\alpha^{n-1}}.$$

Therefore,

$$[J_\lambda]J_\mu p_n = \frac{[J_\lambda]J_\mu J_n - [J_\lambda]J_\mu J_{n-1}p_1 - [J_\lambda]J_\mu J_{n-2}p_2(n-1)\alpha - \ldots - [J_\lambda]J_\mu J_1 p_{n-1}(n-1)!\alpha^{n-2}}{(n-1)!\alpha^{n-1}}.$$

From Proposition 58, we know that

$$[J_\lambda]J_\mu J_n = 0,\ [J_\lambda]J_\mu J_{n-1} = 0,\ [J_\lambda]J_\mu J_{n-2} = 0,\ \ldots,\ [J_\lambda]J_\mu J_2 = 0.$$

Hence,

$$[J_\lambda]J_\mu p_n = -\frac{[J_\lambda]J_\mu p_1 p_{n-1}}{\alpha}.$$

Let a partition $\delta$ be so that $\delta/\mu$ consists of a single box in column $\eta_1$ of $\mu$. Then,

$$\begin{aligned} J_\mu p_1 &= \prod_{s\in\eta_1} \frac{h^*_\mu(s)}{h^*_\delta(s)} \prod_{s\in\mu\setminus\eta_1} \frac{h^\mu_*(s)}{h^\delta_*(s)} J_\delta \\ &= \prod_{s\in\eta_1} \frac{h^*_\mu(s)}{h^*_\delta(s)} \prod_{s\in\xi_1} \frac{h^\mu_*(s)}{h^\delta_*(s)} J_\delta. \end{aligned}$$

Thus,

$$[J_\lambda]J_\mu p_n = -\frac{1}{\alpha}[J_\lambda]J_\mu p_1 p_{n-1} = -\frac{1}{\alpha}\prod_{s\in\eta_1}\frac{h^*_\mu(s)}{h^*_\delta(s)}\prod_{s\in\xi_1}\frac{h^\mu_*(s)}{h^\delta_*(s)}[J_\lambda]J_\delta p_{n-1}.$$

Similarly,

$$[J_\lambda]J_\delta p_{n-1} = \frac{[J_\lambda]J_\delta J_{n-1} - [J_\lambda]J_\delta J_{n-2}p_1 - [J_\lambda]J_\delta J_{n-3}p_2(n-1)(n-2)\alpha - \dots - [J_\lambda]J_\delta J_1 p_{n-2}(n-1)!\alpha^{n-3}}{(n-1)!\alpha^{n-2}}.$$

By Proposition 58,

$$[J_\lambda]J_\delta p_{n-1} = -\frac{1}{\alpha}[J_\lambda]J_\delta p_1 p_{n-2}.$$

Let a partition $\nu$ be so that $\nu/\delta$ consists of a single box in column $\eta_1$ of $\delta$. Hence

$$J_\delta p_1 = \prod_{s\in\eta_1}\frac{h^*_\delta(s)}{h^*_\nu(s)}\prod_{s\in\xi_2}\frac{h^\delta_*(s)}{h^\nu_*(s)}\frac{\alpha}{\alpha+1}J_\nu.$$

Thus

$$[J_\lambda]J_\delta p_{n-1} = -\frac{1}{\alpha}[J_\lambda]J_\delta p_1 p_{n-2} = -\frac{1}{\alpha}\prod_{s\in\eta_1}\frac{h^*_\delta(s)}{h^*_\nu(s)}\prod_{s\in\xi_2}\frac{h^\delta_*(s)}{h^\nu_*(s)}\frac{\alpha}{\alpha+1}[J_\lambda]J_\nu p_{n-2}.$$

$$\begin{aligned}
[J_\lambda]J_\mu p_n &= -\frac{1}{\alpha}\prod_{s\in\eta_1}\frac{h^*_\mu(s)}{h^*_\delta(s)}\prod_{s\in\xi_1}\frac{h^\mu_*(s)}{h^\delta_*(s)}[J_\lambda]J_\delta p_{n-1}\\
&= \frac{1}{\alpha^2}\prod_{s\in\eta_1}\frac{h^*_\mu(s)}{h^*_\delta(s)}\prod_{s\in\eta_1}\frac{h^*_\delta(s)}{h^*_\nu(s)}\prod_{s\in\xi_1}\frac{h^\mu_*(s)}{h^\delta_*(s)}\prod_{s\in\xi_2}\frac{h^\delta_*(s)}{h^\nu_*(s)}\frac{\alpha}{\alpha+1}[J_\lambda]J_\nu p_{n-2}
\end{aligned}$$

Hence

$$[J_\lambda]J_\mu p_n = \frac{(-1)^{n-1}}{\alpha^{n-1}}\prod_{s\in\eta_1}\frac{h^*_\mu(s)}{h^*_\delta(s)}\prod_{s\in\eta_2}\frac{h^*_\delta(s)}{h^*_\nu(s)}\prod_{s\in\eta_3}\frac{h^*_\nu(s)}{h^*_\gamma(s)}\cdots\prod_{s\in\eta_{n-1}}\frac{h^*_\phi(s)}{h^*_\tau(s)}\prod_{s\in\eta_n}\frac{h^*_\tau(s)}{h^*_\lambda(s)}\left(\prod_{k=1}^{n}\prod_{s\in\xi_k}\frac{h^\mu_*(s)}{h^\lambda_*(s)}\right)$$

$$\begin{aligned}
[J_\lambda]J_\mu p_n &= \frac{(-1)^{n-1}}{\alpha^{n-1}}\prod_{s\in\eta_1}\frac{h^*_\mu(s)}{h^*_\delta(s)}\prod_{s\in\eta_1}\frac{h^*_\delta(s)}{h^*_\nu(s)}\frac{\alpha}{\alpha+1}\prod_{s\in\eta_1}\frac{h^*_\nu(s)}{h^*_\gamma(s)}\frac{\alpha}{\alpha+2}\\
&\quad \cdots\prod_{s\in\eta_1}\frac{h^*_\phi(s)}{h^*_\tau(s)}\frac{\alpha}{\alpha+(n-2)}\prod_{s\in\eta_1}\frac{h^*_\tau(s)}{h^*_\lambda(s)}\frac{\alpha}{\alpha+(n-1)}\left(\prod_{k=1}^{n}\prod_{s\in\xi_k}\frac{h^\mu_*(s)}{h^\lambda_*(s)}\right)\\
&= \prod_{t=1}^{n-1}\frac{(-1)}{\alpha+t}\prod_{s\in\eta_1}\frac{h^*_\mu(s)}{h^*_\lambda(s)}\left(\prod_{k=1}^{n}\prod_{s\in\xi_k}\frac{h^\mu_*(s)}{h^\lambda_*(s)}\right) \qquad (3.1)
\end{aligned}$$

Furthermore,

$$[J_\lambda]J_\mu p_n = \prod_{j=1}^{i_1-1}\frac{h_\mu^*(r_j,s_{i_1}+1)}{h_\mu^*(r_{j-1}+1,s_{i_1}+1)+1}\prod_{j=1}^{i_2-1}\frac{h_\delta^*(r_j,s_{i_2}+1)}{h_\delta^*(r_{j-1}+1,s_{i_2}+1)+1}$$

$$\prod_{j=1}^{i_3-1}\frac{h_\nu^*(r_j,s_{i_3}+1)}{h_\nu^*(r_{j-1}+1,s_{i_3}+1)+1}\prod_{j=1}^{i_{n-1}-1}\frac{h_\tau^*(r_j,s_{i_{n-1}}+1)}{h_\tau^*(r_{j-1}+1,s_{i_{n-1}}+1)+1}$$

$$\prod_{j=1}^{i_n-1}\frac{h_\lambda^*(r_j,s_{i_n}+1)}{h_\lambda^*(r_{j-1}+1,s_{i_n}+1)+1}\prod_{j=i_1}^{m_1}\frac{h_*^\mu(r_{i_1-1}+1,s_j)}{h_*^\mu(r_{i_1-1}+1,s_{j+1}+1)+\alpha}$$

$$\prod_{j=i_2}^{m_2}\frac{h_*^\delta(r_{i_2-1}+1,s_j)}{h_*^\delta(r_{i_2-1}+1,s_{j+1}+1)+\alpha}\prod_{j=i_3}^{m_3}\frac{h_*^\nu(r_{i_3-1}+1,s_j)}{h_*^\nu(r_{i_3-1}+1,s_{j+1}+1)+\alpha}$$

$$\ldots\prod_{j=i_{n-1}}^{m_{n-1}}\frac{h_*^\tau(r_{i_{n-1}-1}+1,s_j)}{h_*^\tau(r_{i_{n-1}-1}+1,s_{j+1}+1)+\alpha}\prod_{j=i_n}^{m_n}\frac{h_*^\lambda(r_{i_n-1}+1,s_j)}{h_*^\lambda(r_{i_n-1}+1,s_{j+1}+1)+\alpha}.$$

Hence

$$[J_\lambda]J_\mu p_n = \frac{(-1)^{n-1}}{(\alpha+1)(\alpha+2)\ldots(\alpha+(n-1))}\prod_{j=1}^{i_1-1}\frac{h_\mu^*(r_j,s_{i_1}+1)}{h_\mu^*(r_{j-1}+1,s_{i_1}+1)+1}$$

$$\prod_{j=1}^{i_2-1}\frac{h_\mu^*(r_j,s_{i_2}+1)+1}{h_\mu^*(r_{j-1}+1,s_{i_2}+1)+1+1}\prod_{j=1}^{i_3-1}\frac{h_\mu^*(r_j,s_{i_3}+1)+1+1}{h_\mu^*(r_{j-1}+1,s_{i_3}+1)+1+1+1}$$

$$\ldots\prod_{j=1}^{i_{n-1}-1}\frac{h_\mu^*(r_j,s_{i_{n-1}}+1)+1+1+\ldots+1}{h_\mu^*(r_{j-1}+1,s_{i_{n-1}}+1)+1+1+1+\ldots+1}$$

$$\prod_{j=1}^{i_n-1}\frac{h_\mu^*(r_j,s_{i_n}+1)+1+1+1+\ldots+1}{h_\mu^*(r_{j-1}+1,s_{i_n}+1)+1+1+1+1+\ldots+1}$$

$$\prod_{k=1}^{n}\prod_{j=i_1}^{m_1}\frac{h_*^\mu(r_{i_1-1}+1,s_j)-(k-1)}{h_*^\mu(r_{i_1-1}+1,s_{j+1}+1)+\alpha-(k-1)}$$

$$= \prod_{t=1}^{n-1}\frac{(-1)}{\alpha+t}\prod_{k=1}^{n}\prod_{j=1}^{i_1-1}\frac{h_\mu^*(r_j,s_{i_k}+1)+(k-1)}{h_\mu^*(r_{j-1}+1,s_{i_k}+1)+k}$$

$$\prod_{j=i_1}^{m_1}\frac{h_*^\mu(r_{i_1-1}+1,s_j)-(k-1)}{h_*^\mu(r_{i_1-1}+1,s_{j+1}+1)+\alpha-(k-1)} \tag{3.2}$$

We see that the Equation 3.1 is equivalent to the Equation 3.2.

□

## 3.2 The Coefficient of $J_\lambda$ in the Product $J_\mu p_n$ where $\lambda/\mu$ is a First Row of $n$ Boxes

**Theorem 67.** *Let $\mu = (\mu_1, \mu_2, ..., \mu_l)$ be any partition. Let $\lambda = (\mu_1 + n, \mu_2, ..., \mu_l)$. Algorithm 61 for $J_\mu\, p_n$ calculates the coefficient of $J_\lambda$.*

*Proof.* Since $\mu = (\mu_1, \mu_2, ..., \mu_l)$ and $\lambda = (\mu_1 + n, \mu_2, ..., \mu_l)$ we have

$$\lambda/\mu = (n) = \square\square\square\square\square\square\square\square\square\square.$$

Thus we obtain the coefficient of $J_\lambda$ in the product $J_\mu p_n$ where $\lambda/\mu$ is a first row of $n$ boxes using Theorem 59;

$$[J_\lambda]J_\mu J_n \;=\; \prod_{x\in\xi} \frac{h_*^\mu(x)}{h_*^\lambda(x)}.$$

$$[J_\lambda]J_\mu p_n = \prod_{i=1}^{n-1} \frac{1}{i\alpha+1} \prod_{s\in\xi} \frac{h_*^\mu(s)}{h_*^\mu(s)+\alpha n} = \prod_{i=1}^{n-1} \frac{1}{i\alpha+1} \prod_{s\in\xi} \frac{h_*^\mu(s)}{h_*^\lambda(s)} \tag{3.3}$$

Also, take $\mu = (\mu_1, \mu_2, ..., \mu_l), \nu = (\mu_1 + 1, \mu_2, ..., \mu_l), \psi = (\mu_1 + 1 + 1, \mu_2, ..., \mu_l) = (\mu_1 + 2, \mu_2, ..., \mu_l), ..., \gamma = (\mu_1 + 1 + ... + 1, \mu_2, ..., \mu_l) = (\mu_1 + (n-1), \mu_2, ..., \mu_l), \lambda = (\mu_1 + 1 + 1 + ... + 1, \mu_2, ..., \mu_l) = (\mu_1 + n, \mu_2, ..., \mu_l)$.

$$\begin{aligned}
[J_\lambda]J_\mu p_n \;=\; & \prod_{j=i_1}^{m_1} \frac{h_*^\mu(r_{i_1-1}+1, s_j)}{h_*^\mu(r_{i_1-1}+1, s_{j+1}+1)+\alpha} \prod_{j=i_1}^{m_2} \frac{h_*^\nu(r_{i_1-1}+1, s_j)}{h_*^\nu(r_{i_1-1}+1, s_{j+1}+1)+\alpha} \\
& \prod_{j=i_1}^{m_3} \frac{h_*^\psi(r_{i_1-1}+1, s_j)}{h_*^\psi(r_{i_1-1}+1, s_{j+1}+1)+\alpha} \cdots \prod_{j=i_1}^{m_{n-1}} \frac{h_*^\gamma(r_{i_1-1}+1, s_j)}{h_*^\gamma(r_{i_1-1}+1, s_{j+1}+1)+\alpha} \\
& \prod_{j=i_1}^{m_n} \frac{h_*^\lambda(r_{i_1-1}+1, s_j)}{h_*^\lambda(r_{i_1-1}+1, s_{j+1}+1)+\alpha}
\end{aligned}$$

We obtain the coefficient of $J_\lambda$ in the product $J_\mu p_n$ where $\lambda/\mu$ is a first row of $n$ boxes using Algorithm 61 where $i_1 = 1$ in the following;

$$[J_\lambda]J_\mu p_n = \prod_{j=1}^{m_1} \frac{h_*^\mu(r_0+1,s_j)}{h_*^\mu(r_0+1,s_{j+1}+1)+\alpha} \prod_{j=1}^{m_2} \frac{h_*^\nu(r_0+1,s_j)}{h_*^\nu(r_0+1,s_{j+1}+1)+\alpha}$$

$$\prod_{j=1}^{m_3} \frac{h_*^\psi(r_0+1,s_j)}{h_*^\psi(r_0+1,s_{j+1}+1)+\alpha} \cdots \prod_{j=1}^{m_{n-1}} \frac{h_*^\gamma(r_0+1,s_j)}{h_*^\gamma(r_0+1,s_{j+1}+1)+\alpha}$$

$$\prod_{j=1}^{m_n} \frac{h_*^\lambda(r_0+1,s_j)}{h_*^\lambda(r_0+1,s_{j+1}+1)+\alpha}$$

$$= \prod_{c=1}^{n-1} \frac{1}{(c\alpha+1)} \prod_{k=1}^{n} \prod_{j=1}^{m_1} \frac{h_*^\mu(r_0+1,s_j)+(k-1)\alpha}{h_*^\mu(r_0+1,s_{j+1}+1)+k\alpha} \tag{3.4}$$

Hence the Equations 3.3 and 3.4 are equivalent to each other. □

## 3.3 The Coefficient of $J_\lambda$ in the Product $J_\mu p_n$ where $\lambda/\mu$ is a Row of $n$ Boxes

**Theorem 68.** *Let $\lambda$ and $\mu$ be two Young diagrams whose difference $\lambda/\mu$ is a single row of n boxes. Algorithm 61 for $J_\mu\, p_n$ calculates the coefficient of $J_\lambda$.*

*Proof.* Since

$$[J_\lambda]J_\mu J_n = \prod_{x\in\xi} \frac{h_*^\mu(x)}{h_*^\lambda(x)} \prod_{x\in\eta} \frac{h^*_\mu(x)}{h^*_\lambda(x)}.$$

$$[J_\lambda]J_\mu p_n = \prod_{i=1}^{n-1} \frac{1}{i\alpha+1} \prod_{s\in\xi} \frac{h_*^\mu(s)}{h_*^\lambda(s)} \prod_{k=1}^{n} \prod_{s\in\eta_k} \frac{h^*_\mu(s)}{h^*_\lambda(s)} \tag{3.5}$$

Also, we obtain

$$[J_\lambda]J_\mu p_n = \prod_{c=1}^{n-1} \frac{1}{(c\alpha+1)} \prod_{k=1}^{n} \prod_{j=i_1}^{m_1} \frac{h_*^\mu(r_{i_1-1}+1,s_j)+(k-1)\alpha}{h_*^\mu(r_{i_1-1}+1,s_{j+1}+1)+k\alpha}$$

$$\prod_{j=1}^{i_1-1} \frac{h^*_\mu(r_j,s_{i_1}+1)-(k-1)\alpha}{h^*_\mu(r_{j-1}+1,s_{i_1}+1)+1-(k-1)\alpha} \tag{3.6}$$

Hence the Equations 3.5 and 3.6 are equivalent to each other. □

Consequently, we obtained the formulas of the product $J_\mu\, p_n$ for some cases in this chapter. We showed that the formulas which were obtained using Theorem 59 coincided with the formulas which were obtained using Algorithm 61.

# 4. A NEW COMBINATORIAL IDENTITY FOR CATALAN NUMBERS

Catalan numbers enumerate a diverse collection of disparate mathematical objects which seem unrelated at first impression. For a nonnegative integer $n$, the $n$-th Catalan number, $C_n$, is $\frac{1}{n+1}\binom{2n}{n}$. A standard combinatorial definition is that the Catalan number, $C_n$, is the number of Dyck paths in an $n \times n$ box. A Dyck path in $n \times n$ box is a path starting from the corner $(0,0)$ to the corner $(n,n)$ which stays always weakly below the diagonal (or always weakly above).

For any positive integer $n$ and integer $r$, define the set $U(n,r)$ as in [5]:

$$U(n,r) = \left\{ (u_i) \in \mathbb{N}^{n+1} : \sum_{i=0}^{n} u_i = n \text{ and } \sum_{i=0}^{n} i\,u_i \equiv r \pmod{n+1} \right\}.$$

The following set $V(n)$ appears as $(q^5)$ in Stanley's Catalan Addendum [12]:

$$V(n) = \left\{ (v_i) \in \mathbb{N}^{n} : \sum_{i=1}^{n} v_i = n \text{ and } \sum_{i=1}^{j} v_i \geq j \text{ for all } j = 1, \ldots, n \right\}.$$

For a sequence $w = (w_0, w_1, \ldots, w_n)$ of total $n$, denote the multinomial coefficient $\binom{n}{w_0, w_1, \ldots, w_n}$ by $\binom{n}{w}$.

As in [5], attach to the sets $U(n,r)$ and $V(n)$ generating functions, $\sum_w q^{\binom{n}{w}}$ where $q$ is an indeterminate and the index $w$ runs over the corresponding set. Denote the generating functions by $u(n,r)$ and $v(n)$ respectively.

In [5], Aker and Can conjecture that

**Conjecture** (Conjecture 1.1 in [5])**.** *For a positive integer $n$ and an integer $r$, the generating functions $u(n,r)$ and $v(n)$ coincide.*

We prove this conjecture in Theorem 81 as a direct corollary of a bijection established between the sets $U(n,r)$ and $V(n)$ in Theorem 80.

### 4.1 Preliminaries

**Definition 69.** Let $a = (a_1, \ldots, a_n)$ be a sequence of positive integers and $b_1 \leq b_2 \leq \ldots \leq b_n$ be the non-decreasing rearrangement of $a$ for a given positive integer $n$. Then the sequence $a$ is called a *parking function* of length $n$ if and only if $b_i \leq i$ for $i = 1, \ldots, n$. Denote the set of parking functions of length $n$ by $PF(n)$.

**Theorem 70** ( [13]). *The number of parking functions of length n is* $(n+1)^{n-1}$.

**Lemma 71.** *Every permutation of a parking function is also a parking function.*

This is to say that the symmetric group on $n$ letters $\mathfrak{S}_n$ acts on the set $PF(n)$ of parking functions of length $n$ of cardinality $(n+1)^{n-1}$. If $n = 3$, then the set of parking functions is divided into 5 orbits under the symmetric group action as follows:

$\mathbf{111}$
$\mathbf{112}, 121, 211$
$\mathbf{113}, 131, 311$
$\mathbf{122}, 212, 221$
$\mathbf{123}, 132, 213, 231, 312, 321$

Each row above represents a symmetric group orbit whose first entry is chosen to be non-decreasing. Notice that orbits are parameterized by non-decreasing parking functions. The number of symmetric group orbits inside the set of parking functions, $PF(n)$, is the $n$-th Catalan number, $C_n$.

**Corollary 72.** *The number of non-decreasing positive integer sequences* $(b_1, \ldots, b_n)$ *so that* $1 \leq b_i \leq i$ *for all* $i = 1, \ldots, n$ *is the n-th Catalan number,*

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

The following set (see [1])

$$B(n) = \{(b_i) \in \mathbb{N}^n : 1 \leq b_1 \leq b_2 \leq \cdots \leq b_n \leq n \text{ and } b_i \leq i\}$$

also has cardinality $C_n$. In other words, we can directly produce a bijection between the set of Dyck paths and $B(n)$ as follows:

For a given Dyck path, record the heights of each step. The sequence of heights is an element of the set above. Conversely, given a weakly increasing sequence as above determines a Dyck path.

We can construct a bijection between the sets $B(n)$ and $V(n)$, both of which has cardinality the Catalan number $C_n$.

**Example 73** ( [14])**.** Consider a Dyck path on an $n \times n$ grid and label the horizontal line segments by their height above the $x$-axis plus one. For instance, $B(3) = \{111, 112, 113, 122, 123\}$.

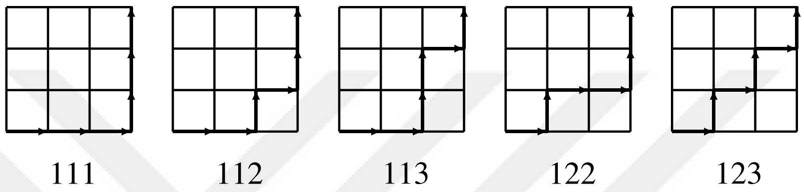



**Figure 4.1** : Valid Paths

On the other hand, use the labels to construct the corresponding $n$-tuple as follows: Count the number of $1s, 2s$ and $3s$ and so on among the labels. Take the counts to form the $n$-tuple $v_1, v_2, \ldots, v_n$, where $v_i$ denotes the number of labels $i$, where $1 \leq i \leq n$.
For instance, consider the labels $1, 1$ and $1$ of the first $3 \times 3$ grid in figure 4.1. Then $v_1 = 3, v_2 = 0, v_3 = 0$. The corresponding 3-tuple is 300. We can see that the desired properties are satisfied:
1) Every $v_i$ is nonnegative.
2) The sum of every partial sum with $j$ summands is $\geq j$.
3) The total sum is 3.
Hence $V(3) = \{300, 210, 201, 120, 111\}$. Notice that there is a one to one correspondence between $B(3)$ and $V(3)$.

## 4.2 The Sets $U(n)$, $U(n,r)$ and the Shift Operator

In this section, we prove that the cardinality of the set $U(n,r)$ is equal to the $n$-th Catalan number, $C_n$.

For a positive integer $n$, the following sets are in bijection:

$$U(n) := \left\{(u_0,\dots,u_n)\in\mathbb{N}^{n+1} : \sum_{i=0}^{n} u_i = n\right\},$$

$$\overline{U}(n) := \left\{(u_0,\dots,u_n)\in(\mathbb{Z}/(n+1)\mathbb{Z})^{n+1} : \sum_{i=0}^{n} u_i = n\right\}.$$

Denote the set of $n$-element subsets of a set $X$ by $\binom{X}{n}$ and the set $\{1,2,3,\dots,n\}$ by $[n]$. For $u=(u_0,u_1,...,u_n)\in U(n)$, define $F : U(n)\to\binom{[2n]}{n}$ as follows:

$$F(u_0,u_1,...,u_n) := \{u_1+1 < u_1+1+u_2+1 < ... < u_1+u_2+...+u_n+n\}.$$

Then,

**Lemma 74.** *The map $F : U(n)\to\binom{[2n]}{n}$ is a bijection. The cardinalities of the sets $U(n)$ and $\overline{U}(n)$ are equal to $\binom{2n}{n}$.*

*Proof.* First, the map $F$ is well–defined: Since $0\le u_1$, we have $1\le u_1+1$. Similarly, $0\le u_i$ implies that for $i=1,\dots,n$,

$$u_1+u_2+\cdots+u_{i-1}+i-1 < u_1+u_2+\cdots+u_i+i.$$

We also have $u_1+u_2+\cdots+u_n+n \le u_0+u_1+u_2+\cdots+u_n+n = n+n = 2n$.

The sequence

$$u_1+1 < u_1+u_2+2 < \cdots < u_1+u_2+\cdots+u_i+i < \cdots < u_1+u_2+\cdots+u_n+n$$

forms an $n$-element subset of the set $[2n]$.

We prove that $F$ is a bijection by providing an inverse function, $G$.

For any $n$-element $a=\{a_1<a_2<...<a_n\}$ subset of $[2n]$, set

$$G(a) := (2n-a_n, a_1-1, a_2-a_1-1, \dots, a_n-a_{n-1}-1).$$

Let $u=G(a)$. Such $u$ lies in $U(n)$; that is, all entries of $u$ are nonnegative and they add up to $n$. Because $1\le a_1$, we have $u_1=a_1-1\ge 0$. Similarly for $i=2,\dots,n$, $a_{i-1}<a_i$, hence $u_i=a_i-a_{i-1}-1\ge 0$. Finally, $a_n\le 2n$ implies that $u_0=2n-a_n\ge 0$.

Also the sum of all terms telescope and cancel each other:

$$2n - a_n + a_1 - 1 + a_2 - a_1 - 1 + \cdots + a_n - a_{n-1} - 1 = 2n - n = n.$$

Clearly, $F$ and $G$ are inverses of each other, hence $F$ is a bijection. $\square$

Let $s$ be the cyclic shift operator on the set $U(n)$: For $(u_0, \ldots, u_n)$, set

$$s(u_0, \ldots, u_n) := (u_1, \ldots, u_n, u_0).$$

The operator $s$ induces an action of $\mathbb{Z}/(n+1)\mathbb{Z}$ on the set $U(n)$.

Define another map $\psi : U(n) \longrightarrow \mathbb{Z}/(n+1)\mathbb{Z}$. For $(u_0, \ldots, u_n) \in U(n)$, set

$$\psi(u_0, \ldots, u_n) := \sum_{i=0}^{n} i\, u_i.$$

**Lemma 75.** *1. For any $u \in U(n)$, $\psi(s(u)) = \psi(u) + 1$.*

*2. Cyclic shift operator $s$ is a fixed-point free automorphism of $U(n)$.*

*3. For any $r \in \mathbb{Z}/(n+1)\mathbb{Z}$, shift operator $s$ takes the set $U(n,r)$ bijectively to $U(n,r+1)$.*

*Proof. 1.* For any $u = (u_0, \ldots, u_n) \in U(n)$,

$$\begin{aligned}\psi(s(u)) &= \sum is(u)_i = \sum iu_{i+1} = \sum_{i=0}^{n}(j-1)u_j = \sum_{i=0}^{n} ju_j - \sum_{i=0}^{n} u_j \\ &= \psi(u) - n \equiv \psi(u) + 1.\end{aligned}$$

*2.* Suppose the automorphism $s$ fixes some $u = (u_0, \ldots, u_n) \in U(n)$, this implies that all $n+1$ coordinates of $u$ are equal. On the other hand, as an element in $U(n)$, sum of the coordinates of $U(n)$ is equal to $n$, which is clearly a contradiction. Therefore the automorphism $s$ is fixed-point free.

*3.* Since it is an automorphism, any restriction of $s$ to a subset of $U(n)$ is a bijection. By (1), the automorphism $s$ maps $U(n,r)$ to $U(n,r+1)$ which shows that the restriction $s : U(n,r) \to U(n,r+1)$ is a bijection. $\square$

**Corollary 76.** *For a positive integer $n$ and an integer $r$, the cardinality of the set $U(n,r)$ is the $n$-th Catalan number, $C_n$.*

*Proof.* Note that $U(n)$ is a disjoint union of $U(n,r)$'s where $r \in \mathbb{Z}/(n+1)$:

$$U(n) = \bigsqcup_{r \in \mathbb{Z}/(n+1)} U(n,r)$$

and

$$|U(n)/\langle s \rangle| = \frac{1}{|\langle s \rangle|}|U(n)| = \frac{1}{n+1}\binom{2n}{n} = |U(n,r)|.$$

$\square$

### 4.3 Necklaces and the Main Result

In this section, we prove the equality of the generating functions $u(n,r)$ and $v(n)$. We first establish a bijection the sets $U(n)/\langle s \rangle$ and $V(n)$, which in return produces a bijection between $U(n,r)$ and $V(n)$. The equality of the generating functions follows as a direct corollary.

Define a *string of pearls* to be a finite sequence of nonnegative integers. Elements of the sequence are called *pearls*, each with an assigned value in the string. For convenience, we allow such a string to be circular. Such a *circular* string is called a *necklace*.

**Definition 77.** Given a string of pearls $A$ labelled sequentially $a_1, a_2, \ldots$, by $\ell(A)$ denote the length of string $A$ and by $|A|$ denote the sum $a_1 + a_2 + \cdots$.

A subsequence $S$ of a string $A$ consisting of consecutive pearls is called a *substring*. Write $S \leq A$. Denote the set of all substrings of $A$ by $Sub(A)$. Then, $Sub(A, \leq)$ is a partially ordered set.

Call a string $B$ a *block* if $b_1 + b_2 + \cdots + b_k \geq k$ for all $k = 1, \ldots, \ell(B)$. *Blocks* of a string $A$ are those substrings which are also blocks. Denote the set of all blocks of $A$ by $Blocks(A)$.

Let $A = (1,0,2,1,0,3)$. For instance, $(2,1,0,3)$ is substring, whereas $(2,1,3)$ is not. The blocks of $A$ are $(1)$, $(2)$, $(2,1)$, $(2,1,0)$, $(2,1,0,3)$, $(1)$ (this is the 1 to the right of 2) and $(3)$. Note that $A$ is not a block.

If a string $A$ has at least one positive pearl, the set of blocks of $A$ is not empty. The partial order $\leq$ on the set of substrings of $A$ induces a partial order on the set of blocks of $A$.

Notice that in the above example, each positive pearl is contained in a unique block of maximal length.

Now fix a necklace $N$ in $U(n)/\langle s\rangle$, i.e. a circular string of $n+1$ nonnegative integers whose sum is $n$. Note that any such necklace contains at least one pearl with label 0.

Fix a clockwise orientation for necklaces. For instance, in the figure is the necklace

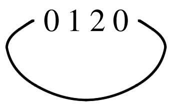



**Figure 4.2** : An example of a necklace

$(0,1,2,0)$ or 0120, which can be equivalently written as 1200, 2001, or as 0012.

**Lemma 78.** *Suppose $B$ is a maximal block of $N$. Then,*

1. *Pearls adjacent to $B$ are labelled* 0.

2. $|B| = \ell(B)$.

*Proof.* Let's analyze the pearls adjacents to the maximal block $B$ in the necklace $N$.

*1.* Let's say the pearl $P$ after $B$ has a label $\geq 1$. That is, $BP$ is a string of pearls, where $B$ is a maximal block and $P \geq 1$.

Then, $|BP| = |B| + |P| \geq \ell(B) + 1 = \ell(BP)$.

Hence $BP$ is a block which contains $B$. This contradicts the maximality of $B$. Reversing the orientation proves the statement for the pearl preceeding the maximal block $B$. So, any pearl next to $B$ is labelled 0.

*2.* Assume that $|B| > \ell(B)$.

We proved that a pearl $P$ adjacent to $B$ is labelled 0. (There must be such a pearl, otherwise $|B| \geq \ell(B) \geq n+1$).

Say $P$ follows $B$. Then $BP$ is a block: Because $|B| \geq \ell(B) + 1$;

$$|BP| = |B| + |P| = |B| \geq \ell(B) + 1 = \ell(BP).$$

Once again, this contradicts the maximality of $B$. Therefore for any maximal block $B$ is stacked by 0's before and after and $|B| = \ell(B)$. $\square$

Being a poset, the set of blocks of the necklace $N$ must have maximal blocks. In fact,

**Lemma 79.** *A necklace $N$ contains a unique maximal block $B$, where $|B| = \ell(B) = n$.*

*Proof.* Lemma 78 implies that the necklace $N$ consists of (possibly several) maximal blocks $B_1,...,B_m$ separated by strings of zeros (Figure 4.3).

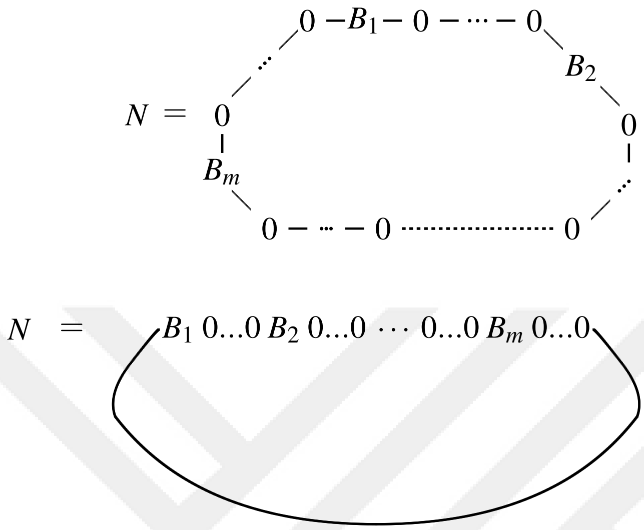



**Figure 4.3** : The necklace $N$ depicted in two different, yet equivalent forms

Note that

- Sum of all pearls $= |B_1| + ... + |B_m| = n$,
- Number of pearls $= \ell(B_1) + ... + \ell(B_m) + \underbrace{m}_{\text{for } m \text{ zeros}} = n+1$.

Therefore,

$$n+1 = \ell(B_1) + ... + \ell(B_m) + m.$$

Because blocks $B_1,\dots,B_m$ are maximal,

$$n+1 = |B_1| + ... + |B_m| + m = n+m.$$

If follows that $m = 1$, i.e. the necklace $N$ contains a unique maximal block $B$, where $|B| = \ell(B) = n$. $\square$

Notice that the maximal block of a necklace is an element of the set

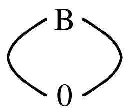
$$V(n) = \left\{ (v_i) \in \mathbb{N}^n : \sum_{i=1}^{n} v_i = n \text{ and } \sum_{i=1}^{j} v_i \geq j \text{ for all } j = 1,...,n \right\}.$$

A direct consequence of the previous lemma is

**Theorem 80.** *The following map is a bijection:*

$$\phi : V(n) \longrightarrow \underbrace{U(n)/\langle s\rangle}_{Necklaces}$$

$$B \longrightarrow \text{The necklace}$$




A direct corollary of the bijection is

**Theorem 81** (Conjecture 1.1 in [5])**.** *For a positive integer $n$ and an integer $r$, the generating functions $u(n,r)$ and $v(n)$ coincide.*

*Proof.* For $v$ in $V(n)$, let $u = \phi(v)$. Then, $\binom{n}{u} = \binom{n}{v}$. By Theorem 80 and Corollary 76,

$$v(n) = \sum_{v\in V(n)} q^{\binom{n}{v}} = \sum_{u\in U(n)/\langle s\rangle} q^{\binom{n}{u}} = \sum_{u\in U(n,r)} q^{\binom{n}{u}} = u(n,r).$$

$\square$

# 5. CONCLUSIONS AND RECOMMENDATIONS

In this thesis, we study to two problems from algebraic combinatorics. The first problem, studied in Chapters 1-3, is to obtain a formula for the product of Jack polynomials and power-sum symmetric polynomials in terms of Jack polynomials. In this direction, we show that algorithm put forward by Sakamoto et al. calculates the expansion coefficients correctly for such a product in a number of special cases. Let $n$ be a nonnegative integer.More specifically, we are able to verify the said algorithm produces the coefficients for the following types of products:

- $J_\mu p_k$ where $\mu$ is arbitrary and $k = 1,2$.
- $J_\mu p_k$ where $\mu = 1^n$ and $k = 1$.
- $J_\mu p_k$ where $\mu = n$ and $k = 1,2,3,4$.
- $J_\mu p_k$ where $\mu = n$ and $k$ is an arbitrary positive integer.
- $J_\mu p_k$ where $\mu = m \times n$ rectangular diagram and $k = 2$.
- The coefficient of $J_\lambda$ in the product $J_\mu p_k$ where $\lambda/\mu$ is a column of $k$ boxes and $\lambda, \mu, k$ are arbitrary.
- The coefficient of $J_\lambda$ in the product $J_\mu p_k$ where $\lambda/\mu$ is a row of $n$ boxes and $\lambda, \mu, k$ are arbitrary.

In Chapter 4, we prove a conjecture by Aker and Can [5] which claims the equality of two generating functions. This is done by constructing a bijection between the underlying sets of the generating functions.

Sakamoto et al. study the action of Virasoro operators $L_n$ on the Fock space, namely on the Jack polynomials. An algorithm to calculate the matrices of a Virasoro operators in Jack basis is given. One possible direction to pursue is to prove the algorithm in this more general context. One benefit of this context is that once the action of Virasoro operators $L_1$ and $L_2$ is known, the effect of all Virasoro operators $L_n$ with $n \geq 3$ can be constructed using the commutation relations:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta m + n, 0,$$

where $c$ is central charge. Here, $m$ and $n$ are arbitrary integers.

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